ON THE BROADENING OF NEUTRAL HELIUM LINES IN STELLAR SPECTRA

MARGARET K. KROGDAHL
Yerkes Observatory and Evanston, Illinois
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ABSTRACT

The broadening of helium lines in early-type stellar spectra is considered here from the point of view of the direct interactions between a proton and a helium atom which were calculated in earlier papers. The theory of such broadening is discussed with a view toward determining, in terms of the physical conditions in the stellar atmosphere, whether it may be expected to be predominantly of the Lorentz-Weisskopf type or of the statistical, or Holtsmark, type. The results of this discussion are applied to the calculation of the broadening of several helium lines for the case of τ Scorpii. It is found that for most of these lines the broadening due to direct interactions is largely of the collisional type and, further, that such a theory appears to be capable of accounting for the equivalent widths within the uncertainties of the observations.

I. REMARKS ON THE PROBLEM OF LINE BROADENING

1. *Introduction.*—The problem of the pressure broadening of helium lines in early-type stellar spectra has been discussed at length by several authors, including Struve, Foster and Douglas, Verweij, and Unsöld.[1] Despite the fact that it was pointed out by Struve as early as 1938 that collisional broadening by ions might play a role in this connection (in particular, with regard to the "filling-in" of the space between λ 4470 and λ 4471), most of the subsequent treatments have been primarily concerned with what might be termed the "statistical" Stark effect, arising from the electric fields present in the atmosphere. However, as has been mentioned in connection with a related problem,[2] the electric fields acting upon absorbing atoms in a stellar atmosphere are caused by the neighboring charged particles, which, moreover, because of the high temperatures, are in a continual state of motion with respect to the absorbing atoms. Considerations such as this suggest that it would also be of interest to discuss the formation of the helium lines in terms of the direct interactions between a helium atom and a neighboring charged particle—specifically, a proton.[3]

2. *On the pressure broadening of spectral lines.*—As has been pointed out in the recent discussions of pressure broadening, the problem can be considered from two points of view. We have, on the one hand, the collision-broadening theory of Lorentz and Weisskopf, in which the time dependence of the perturbation is the principal factor. The wave train of the optical atom is considered to be "interrupted" by collisions with interacting particles; and the resultant intensity distribution, as a function of the pressure and temperature, is obtained from the Fourier analysis of such wave trains. On the other hand, there is the statistical theory originally due to Holtsmark, based on the existence of a direct correspondence between the perturbation and the emitted frequency, the resulting intensity distribution being simply a consequence of the probability with which any given perturbation takes place. It is clear that, in the limiting case of no relative velocity

[1] For an account of some of these investigations see A. Unsöld, *Physik der Sternatmosphären* (Berlin, 1938), § 73; also Struve, *Observatory*, **61**, 53, 1938. More recent papers are those of Foster and Douglas, *M.N.*, **99**, 150, 1939; Verweij, *Proc. Neth. Acad.*, **43**, 1000, 1940; and Unsöld, *Zs.f. Ap.*, **23**, 75, 1944.

[2] Krogdahl, *Ap. J.*, **100**, 333, 1944, and **102**, 64, 1945.

[3] Because of the enormous hydrogen abundance and the high temperatures, protons and electrons will be by far the most numerous charged particles. It is not immediately clear, however, that broadening by electrons can be discussed by the present methods, whose quantum-mechanical justification depends upon the validity of the Franck-Condon principle and the smallness of the De Broglie wave length of the perturbing particles.

between atom and perturber, we should be correct in applying the latter theory.[4] As is well known, however, when there is a relative motion, the central portion of the resulting line is governed by the collision-broadening theory, while the statistical theory must be applied when one considers the wings. Estimates have been made in various ways of the relative ranges of applicability of the two theories.[5]

It is thus apparent that, in order to apply correctly the direct interactions in a calculation of the collisional and/or statistical broadening of any stellar line, it is necessary, first of all, to be able to state the relative importance of the two processes in terms of the fundamental physical properties of the atmosphere, that is, the temperature and the pressure. A simple theory which shows how it is possible to do this and which, at the same time, furnishes a single convenient formula giving the intensity at any point in the line has been developed by Burkhardt.[6] This calculation, which is an attempt to extend the work started by Lenz,[7] is made on the basis of the classical theory; and it may further be noted that it was not intended as anything more than a preliminary theory.

In actual fact, of course, to speak of such matters as the relative importance of collisional and statistical broadening and the "boundary" between them is rather artificial. Ideally, what is needed is a theory which considers the effect of frequency perturbations upon the formation of lines quite generally, without making the very special assumptions which lead to the so-called "limiting" cases. In particular, such a theory should derive these concepts as limiting forms of the general contour and improve our understanding of them rather than try to superpose limiting cases or "smear" one type of distribution over the other. A theory such as this is what was attempted by Lenz, and only very recently extended by Lindholm,[8] who makes a Fourier analysis of the emission frequency with respect to the collision time.

However, inasmuch as the use of concepts like "collision broadening" and "statistical broadening" has proved fruitful in many problems,[9] it is felt that to continue along such lines is sufficiently good for a preliminary study of the broadening of the helium lines. This is especially true in view of other uncertainties in the problem which are unavoidable at this stage (for example, in the observed data, f-values, etc.). Accordingly, in the following section the essential features of Burkhardt's work are presented, together with certain generalizations and extensions (in particular, to the R^{-4} law of interaction) which prove useful in discussing the problem proposed in section 1.

II. CALCULATION OF THE INTENSITY DISTRIBUTION

3. *Introduction of the "collision time."*—It is well known[10] that, if the instantaneous circular frequency of a perturbed oscillator is written in the form

$$\omega_0(t) = \omega_0 + f(t) , \tag{1}$$

a Fourier analysis of the amplitude as a function of the time yields the intensity distribution

$$J(\omega) = \left| \int_{-\infty}^{+\infty} \exp\left[i(\omega_0 - \omega)\, t + i \int^{t} f(t)\, dt \right] dt \right|^2 . \tag{2}$$

[4] See Kuhn and London, *Phil. Mag.*, **18**, 987, 1934.

[5] For a detailed account of these matters see Unsöld, *Vierteljahrsschr. d. Deutsch. Astr. Gesellsch.*, **78**, 213, 1943.

[6] *Zs. f. Phys.*, **115**, 592, 1940.

[7] *Zs. f. Phys.*, **80**, 423, 1933.

[8] *Ark. Mat., Astr. och Fys.*, Vol. **32** A, No. 17, 1945.

[9] See, e.g., Unsöld, *Physik der Sternatmosphären;* also Strömgren, "On the Chemical Composition of the Solar Atmosphere," *Festschrift für Elis Strömgren* (Copenhagen, 1940).

[10] See, e.g., Lenz, *op. cit.*

The problem at hand is to evaluate the above integral for a generalized law of interaction of the Van der Waals type, i.e.,

$$f(t) = \frac{C_p}{r^p(t)} = \Delta\omega \,, \tag{3}$$

where C_p denotes the appropriate Van der Waals constant, measured in c.g.s. units of circular frequency. For convenience we shall make the customary change of variable

$$r^2 = v^2 t^2 + \rho^2 \,, \tag{4}$$

where ρ is the minimum value of r for a given trajectory (assumed to be linear) and v is the relative velocity of the perturbing particle along it.

Consider the phase integral appearing in the exponent, namely,

$$\Theta(t) = \int_{-\infty}^{t} f(t)\, dt = C_p \int_{-\infty}^{t} (v^2 t^2 + \rho^2)^{-p/2} dt \,. \tag{5}$$

TABLE 1

p	$\psi(\beta)$	$\psi(\infty)$
2. . . .	$\tan^{-1}\beta + \frac{\pi}{2}$	π
3. . . .	$\frac{\beta}{(1+\beta^2)^{1/2}} + 1$	2
4. . . .	$\frac{\beta}{2(1+\beta^2)} + \frac{1}{2}\tan^{-1}\beta + \frac{\pi}{4}$	$\frac{\pi}{2}$
6. . . .	$\frac{\beta}{4(1+\beta^2)^2} + \frac{3\beta}{8(1+\beta^2)} + \frac{3}{8}\tan^{-1}\beta + \frac{3\pi}{16}$	$\frac{3\pi}{8}$

Let

$$\beta = \frac{vt}{\rho}\,; \qquad d\beta = \frac{v\,dt}{\rho}\,. \tag{6}$$

Then $\Theta(t)$ becomes

$$\Theta(\beta) = \frac{C_p}{v\rho^{p-1}} \int_{-\infty}^{\beta} (1+\beta^2)^{-p/2} d\beta = \frac{C_p}{v\rho^{p-1}} \psi_p(\beta)\,. \tag{7}$$

The functions $\psi_p(\beta)$ are given in Table 1 for those values of p which are most important in practice. Now, the fundamental assumption made by Lenz and Burkhardt (and later also by Lindholm) is that the function $\psi_p(\beta)$, describing the phase change during the collision, can be adequately represented in the vicinity of $\beta = 0$ by a straight line, whose slope we shall call γ_p. This means that we are replacing the infinitely long collision, in which the frequency change varies according to equation (3), by one of finite duration, in which the frequency change is constant.

The functions $\psi_p(\beta)$ are plotted in Figure 1, together with the straight lines which appear to represent them best. For $p = 6$ the straight line drawn has the slope 0.76, which was adopted by Lenz and Burkhardt in their calculations.[11] It is obvious that

[11] Lindholm (*op. cit.*) feels that these slopes should be lower, which would, of course, increase the asymmetry to some extent.

the straight-line approximation is worse, the lower the value of p. From the figures we find the following values of γ:

$$\left.\begin{array}{llll} p=2\,, & \gamma\simeq 0.54\,; & p=4\,, & \gamma\simeq 0.73\,; \\ p=3\,, & \gamma\simeq 0.69\,; & p=6\,, & \gamma\simeq 0.76\,. \end{array}\right\} \tag{8}$$

Accordingly, $\Theta(\beta)$ takes the form

$$\Theta(\beta)=\left\{\begin{array}{ll} 0 & \beta<-\frac{1}{2}\frac{\psi(\infty)}{\gamma}\,, \\ \frac{C_p}{v\rho^{p-1}}\left[\gamma\beta+\frac{1}{2}\psi(\infty)\right] & -\frac{1}{2}\frac{\psi(\infty)}{\gamma}<\beta<+\frac{1}{2}\frac{\psi(\infty)}{\gamma}\,, \\ \frac{C_p}{v\rho^{p-1}}\psi(\infty) & \beta>+\frac{1}{2}\frac{\psi(\infty)}{\gamma}\,. \end{array}\right\} \tag{9}$$

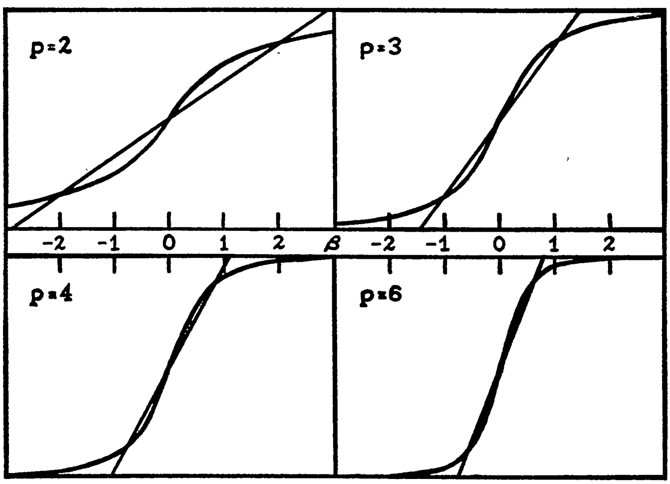

FIG. 1.—The phase functions $\psi_p(\beta)$ and their straight-line approximations. The abscissa in each case is β; the ordinates are, of course, all different.

In terms of t this becomes

$$\Theta(t)=\left\{\begin{array}{ll} 0 & t<-\frac{1}{2}\tau_s\,, \\ \frac{C_p}{v\rho^{p-1}}\gamma\frac{v}{\rho}\left(t+\frac{1}{2}\tau_s\right) & -\frac{1}{2}\tau_s<t<+\frac{1}{2}\tau_s\,, \\ \frac{C_p}{v\rho^{p-1}}\psi(\infty) & t>+\frac{1}{2}\tau_s\,, \end{array}\right\} \tag{10}$$

where the "collision time" τ_s is defined by

$$\tau_s=\frac{\rho}{v}\frac{\psi_p(\infty)}{\gamma}=k_p\frac{\rho}{v}\,. \tag{11}$$

The values of k_p are

$$\left.\begin{array}{ll} k_2=\frac{\pi}{0.54}=5.8\,, & k_4=\frac{\pi}{2\times 0.73}=2.15\,, \\ k_3=\frac{2}{0.69}=2.9\,, & k_6=\frac{3\pi}{8\times 0.76}=1.55\,. \end{array}\right\} \tag{12}$$

The constant value of $\Delta\omega$, which acts during the finite time τ_s, thus is

$$\Delta\omega = \frac{\gamma C_p}{\rho^p}, \tag{13}$$

instead of the value $\Delta\omega = C_p/r^p$ for an infinite time.

With these changes the expression for $J(\omega)$ (eq. [2]) breaks up into two types of integrals:

$$J(\omega) = \left| \sum_i \int_{\tau_{f,i}} e^{i(\omega_0-\omega)t} dt + \sum_j \int_{\tau_{s,j}} e^{i(\omega_0-\omega+\Delta\omega_j)t} dt \right|^2, \tag{14}$$

the former being extended over the various intervals of time, τ_f, during which the atom is not being disturbed, and the latter over the intervening collision times, τ_s, during which there is a constant frequency change, $\Delta\omega$.

Under suitable conditions, that is, if the phase changes produced during the collision times are large enough, the above expression can be decomposed into the sum of the squares of the individual terms.[12] It is customary to exclude temporarily all the "weak" collisions by adopting the Weisskopf condition that $\Theta \gtrsim 1$.[13] (The effect of the weak collisions will be considered in a later paragraph.) With this condition we have

$$J(\omega) = J_f(\omega) + J_s(\omega), \tag{15}$$

where

$$J_f(\omega) = \sum_i \left| \int_{\tau_{f,i}} e^{i(\omega_0-\omega)t} dt \right|^2 \tag{16}$$

and

$$J_s(\omega) = \sum_j \left| \int_{\tau_{s,j}} e^{i(\omega_0-\omega+\Delta\omega)t} dt \right|^2. \tag{17}$$

4. *The symmetric portion of the line.*—The first of these terms, $J_f(\omega)$, simply represents the Fourier analysis of an interrupted, but otherwise undisturbed, wave train—which is precisely the idea underlying the Lorentz-Weisskopf theory of collision broadening. As is well known, we have

$$\int_{\tau_{f,i}} e^{i(\omega_0-\omega)t} dt = \frac{e^{i(\omega_0-\omega)\tau_{f,i}} - 1}{i(\omega_0-\omega)}, \tag{18}$$

whence

$$J_f(\omega) = \sum_i 2\, \frac{1-\cos(\omega_0-\omega)\,\tau_{f,i}}{(\omega_0-\omega)^2}. \tag{19}$$

In order to average over all types of collisions we must define the quantity $\bar{\tau}_f$, the mean duration of an undisturbed wave train, by the equation

$$\bar{\tau}_f = \bar{\tau} - \bar{\tau}_s. \tag{20}$$

The quantity $\bar{\tau}$ is the mean time between collisions, and we shall see later how to determine the average $\bar{\tau}_s$. Upon multiplying the right-hand side of equation (19) by

[12] For a detailed discussion of this question see Burkhardt's paper.

[13] In his dissertation, *Über die Verbreiterung und Verschiebung von Spektrallinien* (Uppsala, 1942), Lindholm has shown how this type of assumption can be improved.

$(1/\bar{\tau}_f)e^{-\tau_f/\bar{\tau}_f}$, a factor giving the relative number of undisturbed wave trains of length τ_f, and, integrating over τ_f, we obtain, finally,

$$J_f(\omega) = \frac{c_f}{(\omega_0 - \omega)^2 + (1/\bar{\tau}_f)^2} . \tag{21}$$

This is the usual dispersion-type formula obtained for collision broadening. The proportionality factor c_f is to be determined later. As usual, the mean time between collisions is given by the relation

$$\frac{1}{\bar{\tau}} = \pi N \bar{v} \rho_0^2 , \tag{22}$$

the relative velocity $\bar{v}$ by

$$\bar{v} = \sqrt{\frac{8RT}{\pi} \frac{\mu_1 + \mu_2}{\mu_1 \mu_2}} , \tag{23}$$

and ρ_0 is the "effective optical radius," which, according to the Weisskopf condition, is

$$\rho_0 = \left[\frac{C_p \psi_p(\infty)}{\bar{v}} \right]^{1/(p-1)} . \tag{24}$$

5. *The asymmetric portion of the line.*—The second term, $J_s(\omega)$, represents the Fourier analysis of those parts of the wave train during which the phase change Θ is increasing linearly with the time and there is a perturbation of constant amount $\Delta\omega$ acting. This term leads to the asymmetric portion of the line and corresponds closely to the Holtsmark or "statistical" type of broadening. Indeed, the Fourier analysis of wave trains, in which the phase change is represented by an infinite straight line, must lead to the Holtsmark expression;[14] for the infinity of the line implies that the configuration of particles is stationary—which is precisely the assumption underlying the formulation of the statistical theory.

Now, $J_s(\omega)$ can be written in the form (cf. eq. [19])

$$J_s(\omega) \propto \sum_i \frac{1 - \cos(\omega_i' - \omega)\,\tau_{s,i}}{(\omega_i' - \omega)^2} , \tag{25}$$

where

$$\omega_i' = \omega_0 + \Delta\omega_i . \tag{26}$$

This expression must now be averaged over all lengths of collision τ_s, that is, over all values of ρ and v (see eq. [11]).

Let us consider, first, all collisions having a given value of ρ, that is, we average, first, over the velocity distribution. This is taken from Fowler,[15] and may be stated as follows: Of the particles which pass, per second, at a distance ρ, the fraction having velocities between v and $v + dv$ is proportional to

$$v\, e^{-(v/v_0)^2} v^2 dv , \tag{27}$$

where

$$v_0 = \frac{\sqrt{\pi}}{2} \bar{v} . \tag{28}$$

[14] For a derivation of this formula see Unsöld, *Vierteljahrschr. d. Deutsch. Astr. Gesellsch.*, **78**, 213, 1943.

[15] *Statistical Mechanics* (Cambridge, 1936).

Hence the contribution to $J_s(\omega)$ of all collisions having a certain value of ρ is

$$j_s(\rho, \omega) \propto \int_0^\infty \frac{1-\cos(\omega'-\omega)\left(\frac{v_0}{v}\right)\tau_{s,0}}{(\omega'-\omega)^2} v e^{-(v/v_0)^2} v^2 dv , \tag{29}$$

where

$$\tau_{s,0} = \frac{2k_p}{\sqrt{\pi}} \frac{\rho}{\bar{v}} . \tag{30}$$

Burkhardt shows that the integral in equation (29) can be very well approximated by the dispersion formula, thus

$$j_s(\rho, \omega) \propto \frac{c_s}{(\omega'-\omega)^2 + 2/\tau_{s,0}^2} , \tag{31}$$

where c_s is a proportionality factor to be determined later. Thus the contribution to the line form of all collisions having the parameter ρ is a dispersion-curve of half-width $\sqrt{2}/\tau_{s,0} = (\sqrt{2\pi}\,\bar{v})/2k_p\rho$, which is displaced from the position ω_0 by the amount $\Delta\omega = \gamma C/\rho^p$.

The resultant value of $J_s(\omega)$ is obtained by averaging, in an appropriate manner, overall the j_s distributions for all values of ρ between 0 and ρ_0.[16] That is, the distribution $j_s(\rho, \omega)$ must first be multiplied by a weight factor giving the relative number of collisions having the parameter ρ, and then be integrated from 0 to ρ_0. Thus we shall have

$$J_s(\omega) = c_s \int_0^{\rho_0} \frac{W(\rho)\, d\rho}{[\omega'(\rho) - \omega]^2 + [\sqrt{2}/\tau_{s,0}(\rho)]^2} . \tag{32}$$

To obtain the function $W(\rho)$, let the probability that a perturbing particle has a parameter between ρ and $\rho + d\rho$ be $2\pi\rho d\rho$. Then the probability that the nearest perturbing particle has a parameter between ρ and $\rho + d\rho$ must be $2\pi\rho d\rho$ times the probability that no particle is within the distance ρ, or

$$W(\rho)\, d\rho = K\rho\, d\rho \times e^{-(4\pi/3)N\rho^3} . \tag{33}$$

N is the number of perturbing particles per cubic centimeter, and the constant K is found from the normalizing condition to be

$$K = \frac{3}{(-\frac{1}{3}, y)!} \left(\frac{4\pi}{3} N\right)^{2/3} , \tag{34}$$

where

$$y = \frac{4\pi}{3} N \rho_0^3 \tag{35}$$

and

$$(-\tfrac{1}{3}, y)! = \int_0^y e^{-t} t^{-1/3} dt = \tfrac{3}{2} y^{2/3} (1 - \tfrac{2}{5} y + \tfrac{1}{8} y^2 + \ldots) . \tag{36}$$ [17]

[16] It will be recalled that larger values of ρ were excluded by the Weisskopf condition. Only a slight error is produced in the final contour by neglecting these values in calculating the asymmetric portion of the line, since their only effect occurs in the center of the line, where the collision broadening is the dominant factor in any case (cf. Fig. 3). The normalization of the asymmetric portion will, however, be affected.

[17] See Jahnke and Emde, *Funktionentafeln* (Leipzig and Dover, 1943), p. 22.

In many practical cases, y is a small quantity compared to unity, and the following approximations are sufficient:

$$(-\tfrac{1}{3}, y)! \simeq \tfrac{3}{2} y^{2/3}, \tag{37}$$

$$K \simeq \frac{2}{\rho_0^2}. \tag{38}$$

It may be noted that, by defining $W(\rho)$ in this fashion, the quantity

$$i_s(\rho) W(\rho)\, d\rho = W(\rho)\, d\rho \int_{-\infty}^{+\infty} j_s(\rho, \omega)\, d\omega = \text{const} \times \rho W(\rho)\, d\rho\,, \tag{39}$$

which represents the total intensity contributed by collisions having the parameter ρ, has been put equal (apart from the normalizing factor) to the statistical probability for ρ, which leads to the Holtsmark formula. Thus the essential difference between the asymmetric distribution obtained here and the Holtsmark distribution is that in the latter the relation between the displacement in the line and the value of ρ is strictly one to one, while in the present case a collision at the distance ρ can produce intensity anywhere in the spectrum. Thus, for example, a negative wing is to be expected for the asymmetric distribution. In the outer portions of the positive wing, however, the half-width of the $j_s(\rho, \omega)$ distributions becomes very small with respect to their displacements $\Delta\omega$, and thus we do approach the one-to-one relation and the Holtsmark distribution.

The integration indicated in equation (32) will be discussed in section 7.

6. *The normalization of* J(ω).—To normalize the final intensity distribution properly, we must add the symmetric and asymmetric distributions and normalize their sum to unity. Thus one condition on the constant factors c_f and c_s (eqs. [21] and [32], respectively) is

$$\int_{-\infty}^{+\infty} (J_f + J_s)\, d\omega = 1\,. \tag{40}$$

Now,

$$I_f = \int_{-\infty}^{+\infty} J_f(\omega)\, d\omega = c_f \bar{\tau}_f \pi\,, \tag{41}$$

and

$$I_s = \int_{-\infty}^{+\infty} J_s(\omega)\, d\omega = c_s \int_0^{\rho_0} \int_{-\infty}^{+\infty} j_s(\rho, \omega) W(\rho)\, d\rho d\omega = c_s \sqrt{2\pi}\, \bar{\tau}_s\,, \tag{42}$$

where the mean collision time, $\bar{\tau}_s$, is given by

$$\bar{\tau}_s = k_p \frac{1}{\bar{v}} \int_0^{\rho_0} \rho W(\rho)\, d\rho = k_p \frac{\bar{\rho}}{\bar{v}}. \tag{43}$$

The quantity $\bar{\rho}$ as defined in this way has the value

$$\bar{\rho} = -\frac{e^{-y} - 1}{(-\tfrac{1}{3}, y)!} \left(\frac{4\pi}{3} N\right)^{-1/3} \tag{44}$$

or, for small y,

$$\bar{\rho} \simeq \tfrac{2}{3} \rho_0\,. \tag{45}$$

Thus, in terms of the constants c_f and c_s, equation (40) becomes

$$1 = c_f \bar{\tau}_f \pi + c_s \sqrt{\frac{2}{\pi}}\, \bar{\tau}_s\,. \tag{46}$$

The second condition on c_f and c_s is that

$$\frac{I_s}{I_f}=\frac{\bar{\tau}_s}{\bar{\tau}_f}, \tag{47}$$

whence, combining equations (46) and (47),

$$c_f=\frac{1}{\pi\bar{\tau}} \quad \text{and} \quad c_s=\frac{1}{\sqrt{2\pi}\,\bar{\tau}}. \tag{48}$$

When these values for c_f and c_s are used in equations (21) and (32), the resultant intensity distribution will be normalized to unity.

An important quantity appearing in the above calculation is the ratio $\bar{\tau}_s/\bar{\tau}_f$. By means of this ratio the relative contributions of the collision and asymmetric broadening to the total intensity can be readily calculated. Because the *sum* of the intensity distributions has been normalized to unity, we may just as well consider the ratio $\bar{\tau}_s/\bar{\tau}$, which shows in a simple way how the relative importance of the asymmetric portion of a given line is uniquely determined by the temperature and pressure of the atmosphere, for we have from equations (43) and (22)

$$\frac{\bar{\tau}_s}{\bar{\tau}}=\pi N\bar{v}\rho_0^2 k_p\frac{\bar{\rho}}{\bar{v}}. \tag{49}$$

For small y this reduces simply to

$$\frac{\bar{\tau}_s}{\bar{\tau}}=\frac{2\pi}{3}k_p\frac{NC_p^{3/(p-1)}}{\bar{v}^{3/(p-1)}}; \tag{50}$$

for large y the relation will differ slightly but is still dependent explicitly on N and $\bar{v}$. Thus we have the result (the general nature of which was to be expected) that the relative contribution to the atomic absorption coefficient by the asymmetric broadening is directly proportional to the density of perturbing particles and inversely proportional to the $3/2(p-1)$ power of the temperature.

7. *The calculation of the asymmetric distribution* (32).—The expression for $J_s(\omega)$ becomes, in full,

$$J_s(\omega)=c_sK\int_0^{\rho_0}\frac{\rho\,e^{-(4\pi/3)N\rho^3}d\rho}{(\omega_0+\Delta\omega-\omega)^2+(\sqrt{2}/\tau_{s,0})^2}. \tag{51}$$

In the following, for the sake of generality, we may let

$$\left.\begin{array}{llll}\omega_0-\omega=d, & \Delta\omega=-\Delta=\dfrac{\gamma C}{\rho^p}, & \Delta_0=\dfrac{\gamma|C|}{\rho_0^p} & \text{if } C \text{ is negative};\\ \omega-\omega_0=d, & \Delta\omega=\Delta=\dfrac{\gamma C}{\rho^p}, & \Delta_0=\dfrac{\gamma C}{\rho_0^p} & \text{if } C \text{ is positive}.\end{array}\right\} \tag{52}$$

The formal results of the two cases are the same; if C is negative, the asymmetry is to the red; and if C is positive, it is to the violet. Thus,

$$\rho^p=\frac{\gamma C}{\Delta};\qquad \rho=(\gamma C)^{1/p}\Delta^{-1/p};\,d\rho=-\frac{1}{p}(\gamma C)^{1/p}\Delta^{-(p+1)/p}d\Delta. \tag{53}$$

Making this change of variable, we find for $\tau_{s,0}$ the value

$$\tau_{s,\,0}=\frac{2k_p}{\sqrt{\pi}\;\bar{v}}\,(\gamma C)^{1/p}\Delta^{-1/p}\,, \tag{54}$$

whence it readily follows that

$$\frac{2}{\tau^2_{s,\,0}}=\frac{\Delta^{2/p}}{\epsilon^2}\,, \tag{55}$$

where, by definition,

$$\frac{1}{\epsilon^2}=\frac{\pi\bar{v}^2}{2k^2_p\,(\gamma C)^{2/p}}\,. \tag{56}$$

Upon substituting equations (52), (53), and (55) in equation (51), we obtain, finally,

$$J_s(d) = c_s K\frac{1}{p}(\gamma C)^{2/p}\int_{\Delta_0}^{\infty}\frac{e^{-(4\pi/3)N(\gamma C/\Delta)^{3/p}}d\Delta}{[(d-\Delta)^2+\Delta^{2/p}/\epsilon^2]\,\Delta^{1+2/p}}\,. \tag{57}$$

It is useful in the following discussion to measure the displacements d and Δ in units of $1/\epsilon^{p/(p-1)}$, that is, we introduce the dimensionless quantities b and β, where

$$d=\frac{b}{\epsilon^{p/(p-1)}}\qquad\text{and}\qquad\Delta=\frac{\beta}{\epsilon^{p/(p-1)}}\,. \tag{58}$$

With these substitutions, equation (57) becomes

$$J_s(b) = c_s K\frac{1}{p}(\gamma C)^{2/p}\epsilon^{(2p+2)/(p-1)}\int_{\mu}^{\infty}\frac{e^{-y(\mu/\beta)^{3/p}}d\beta}{[(b-\beta)^2+\beta^{2/p}]\,\beta^{1+2/p}}\,, \tag{59}$$

where we have written

$$\mu_p=\left(\frac{2}{\pi}\right)^{p/(2[p-1])}\,; \tag{60}$$

and it will be recalled that

$$y=\frac{4\pi}{3}N\rho_0^3\,. \tag{35}$$

It is immediately apparent from equation (59) that, for all lines having $y << 1$, the dependence of $J_s(b)$ on b is the same, apart from a multiplying factor,

$$A = c_s K\frac{1}{p}(\gamma C)^{2/p}\epsilon^{(2p+2)/(p-1)}\,. \tag{61}$$

(This remark applies, of course, only to lines having the same value of p.) Moreover, the integral depends on the constant of interaction C_p only through the quantity y in the exponential term, the maximum range of which is only from 1 to e^{-y}. This fact greatly simplifies the calculation of the line forms, as for such cases the integral can be evaluated once and for all for a continuous set of b-values. Accordingly, we can write

$$J_s(b) = A\,\mathfrak{J}_s(b)\,, \tag{62}$$

in which A is a factor depending only on the particular line, and

$$\mathfrak{J}_s(b) = \int_{\mu}^{\infty}\frac{e^{-y(\mu/\beta)^{3/p}}d\beta}{[(b-\beta)^2+\beta^{2/p}]\,\beta^{1+2/p}} \tag{63}$$

is a function of b which is the same for all lines, provided that their value of y is small. It must be noted, however, that each line has its own scale factor, since the constant C_p occurs in the unit $1/\epsilon^{p/(p-1)}$. Thus in different lines the same value of b will correspond to different values of the distance from the line center, d.

In order to evaluate the integral $\mathfrak{J}_s(b)$, we make, following Burkhardt,[18] the approximation that the function

$$f(b-\beta)=\frac{1}{(b-\beta)^2+\beta^{2/p}} \tag{64}$$

(which is really the dispersion distribution [31]) can be adequately represented beyond $|b-\beta|=\frac{3}{2}\beta^{1/p}$ by the inverse-square function, and between $|b-\beta|=\frac{3}{2}\beta^{1/p}$ and $|b-\beta|=0$ by the straight line passing through the points $(0, \beta^{-2/p})$ and $(\beta^{1/p}, \beta^{-2/p}/2)$. Thus, we let[19]

$$\left.\begin{aligned} f(b-\beta)&=\frac{1}{(b-\beta)^2} \quad &&\text{for} \quad |b-\beta|>\tfrac{3}{2}\beta^{1/p} \\ \text{and}\quad & && \\ f(b-\beta)&=\frac{1}{\beta^{2/p}}-\frac{1}{2}\frac{1}{\beta^{3/p}}|b-\beta| \quad &&\text{for} \quad |b-\beta|\leq\tfrac{3}{2}\beta^{1/p}. \end{aligned}\right\} \tag{65}$$

Accordingly, the integration for $\mathfrak{J}_s(b)$ breaks up into three parts:

$$\mathfrak{J}_s(b)\simeq P+Q+R\,, \tag{66}$$

where

$$P=\int_{\mu}^{\beta_1}\exp\left[-y\left(\frac{\mu}{\beta}\right)^{3/p}\right]\beta^{-(p+2)/p}(b-\beta)^{-2}d\beta\,, \tag{67}$$

$$Q=\int_{\beta_1}^{\beta_2}\left(\frac{1}{\beta^{2/p}}-\frac{1}{2}\frac{1}{\beta^{3/p}}|b-\beta|\right)e^{-y(\mu/\beta)^{3/p}}\beta^{-(p+2)/p}d\beta\,, \tag{68}$$

and

$$R=\int_{\beta_2}^{\infty}\exp\left[-y\left(\frac{\mu}{\beta}\right)^{3/p}\right]\beta^{-(p+2)/p}(b-\beta)^{-2}d\beta\,. \tag{69}$$

The limits of integration β_1 and β_2 vary with the particular value of b; they are the two solutions of the equation

$$b-\beta=\mp\tfrac{3}{2}\beta^{1/p}\,. \tag{70}$$

Figure 2 shows β plotted as a function of b, for several values of p. When, for any value of p, the curves cross below the lines $\beta=\mu_p$, the value μ is to be used.

When the values of β_1 and β_2 are known, for any particular value of b, they can be used in the expressions for P, Q, and R which are summarized in Table 2. Now the integral Q becomes, after some minor transformations,

$$\left.\begin{aligned} Q=\frac{p}{3}\mu^{-4/p}\left(\frac{1}{y}\right)^{4/3}[(\tfrac{1}{3},u_1)!-(\tfrac{1}{3},u_2)!]+\frac{p}{6}\mu^{(p-5)/p}\left(\frac{1}{y}\right)^{(5-p)/3}\left[\left(\frac{2-p}{3},u_1\right)!\right.\\ \left.+\left(\frac{2-p}{3},u_2\right)!-2\left(\frac{2-p}{3},u_b\right)!\right]-\frac{p}{6}\mu^{-5/p}\left(\frac{1}{y}\right)^{5/3}b[(\tfrac{2}{3},u_1)!\\ +(\tfrac{2}{3},u_2)!-2(\tfrac{2}{3},u_b)!]\,, \end{aligned}\right\} \tag{71}$$

[18] See, in particular, Fig. 4 of Burkhardt's paper.

[19] An error of 2 per cent in the normalization of the dispersion distribution (64) or (31) is introduced in replacing it by eq. (65).

where

$$u = y\left(\frac{\mu}{\beta}\right)^{3/p}, \tag{72}$$

and

$$\left.\begin{aligned} u_b &= y\left(\frac{\mu}{b}\right)^{3/p} && \text{if } b > \mu, \\ u_b &= y && \text{if } b \le \mu. \end{aligned}\right\} \tag{73}$$

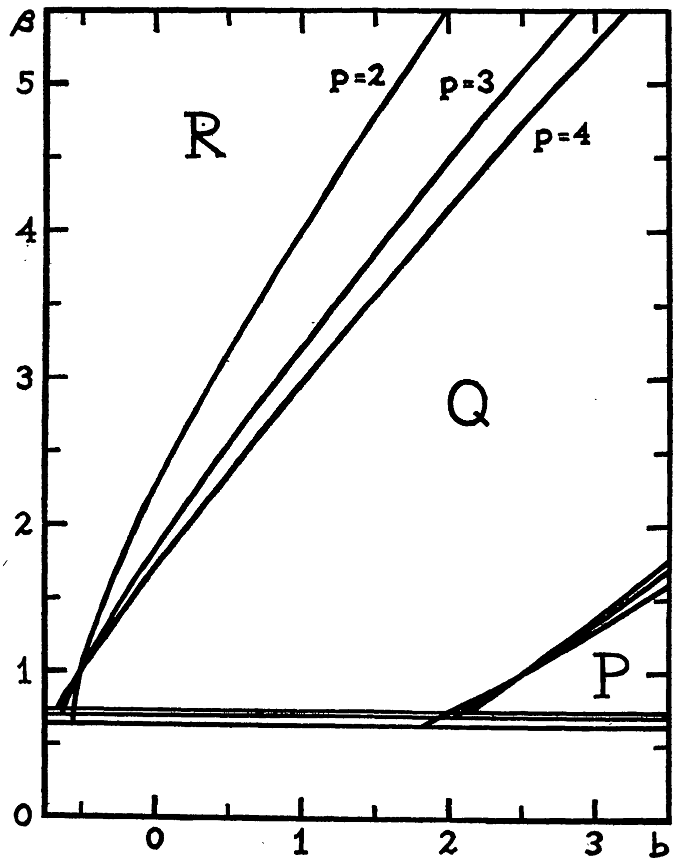

FIG. 2.—The limits of integration β_1 (*lower curves*) and β_2 (*upper curves*) as a function of b. The horizontal lines represent the different values of μ_p, the lower limit of integration. The areas marked P, Q and R show the regions in which the integrals P, Q, and R (eqs. [67]–[69]) are to be used.

Expressions of the type $(x, u)!$ denote the incomplete gamma function (see n. 17)

$$(x, u)! = \int_0^u e^{-t} t^x dt. \tag{74}$$

Unless p is of the form $2 + 3n$, where n is a positive integer, the incomplete gamma functions occurring in the second term of Q may be expanded in terms of u_1, etc. For most of the practical cases, namely, those in which y and therefore u are very small compared to unity, the first terms in the expansions will suffice, and we obtain, finally

$$\left.\begin{aligned} Q_{y<<1} = \frac{p}{4}\left[\beta_1^{-4/p} - \beta_2^{-4/p}\right] + \frac{p}{2(5-p)}\left[\beta_1^{-(5-p)/p} + \beta_2^{-(5-p)/p} - 2\left\{\begin{matrix} b \\ \mu \end{matrix}\right\}^{-(5-p)/p}\right] \\ - \frac{p}{10}\, b\left[\beta_1^{-5/p} + \beta_2^{-5/p} - 2\left\{\begin{matrix} b \\ \mu \end{matrix}\right\}^{-5/p}\right]. \end{aligned}\right\} \tag{75}$$

Equation (75) is the expression for Q to be used when $p = 2, 3, 4, 6$.

With regard to the integration of P, it is clear from Figure 2 that in this region b is always greater than β. Hence we may make the expansion,

$$(b-\beta)^{-2}=b^{-2}\left(1+2\frac{\beta}{b}+3\frac{\beta^2}{b^2}+4\frac{\beta^3}{b^3}+\dots\right), \tag{76}$$

and use as many terms as prove necessary. With this substitution and the change of variable (72), P becomes

$$\left.\begin{aligned}P=&\frac{p}{3}\,b^{-2}\mu^{-2/p}y^{-2/3}[(-\tfrac{1}{3},\,y)\,!-(-\tfrac{1}{3},\,u_1)\,!]\\&+\frac{2p}{3}\,b^{-3}\mu^{(p-2)/p}\left(\frac{1}{y}\right)^{(2-p)/3}\left[\left(-\frac{p+1}{3},\,y\right)!-\left(-\frac{p+1}{3},\,u_1\right)!\right]+\dots\end{aligned}\right\} \tag{77}$$

If y is small, then for all values of p which are not expressible in the form $p=3n-1$, the first two terms of equation (77) become

$$P_{y<<1}=\frac{p}{2}\,b^{-2}\,(\mu^{-2/p}-\beta_1^{-2/p})+\frac{2p}{p-2}\,b^{-3}\,(\beta_1^{(p-2)/p}-\mu^{(p-2)/p})+\dots \tag{78}$$

The third and higher terms are usually not necessary. For $p=2$ the second term of equation (78) must be replaced by the expression

$$2d^{-3}\log_e\left(\frac{\beta_1}{\mu}\right). \tag{79}$$

Finally, it should be noted that $P=0$ if $b<\mu+(3/2)\mu^{1/p}$ (cf. Fig. 2 and eq. [70]).

For the integration of R we note that in the positive line wing and in a small portion of the negative-line wing, β is always greater than $|b|$. For such cases we may write

$$(\beta-b)^{-2}=\beta^{-2}+2b\beta^{-3}+3b^2\beta^{-4}+\dots, \tag{80}$$

and, proceeding exactly as in the case of P, we find that

$$R_{y<<1}=\frac{p}{2p+2}\,\beta_2^{-(2p+2)/p}+\frac{2p}{3p+2}\,b\beta_2^{-(3p+2)/p}+\frac{3p}{4p+2}\,b^2\beta_2^{-(4p+2)/p}+\dots \tag{81}$$

In the major portion of the negative wing (where R is, incidentally, quite small), R may be conveniently transformed by making the substitution

$$\beta=-b\,\frac{x}{1-x}. \tag{82}$$

Thus,

$$R=(-b)^{-(2p+2)/p}\int_{\mu/\mu-b}^{1}\exp\left[-y\left(\frac{\mu}{-b}\,\frac{1-x}{x}\right)^{3/p}\right]\left(\frac{1-x}{x}\right)^{1+2/p}dx. \tag{83}$$

This integral may be fairly easily evaluated by graphical means. If $y\ll 1$, the exponential term is very nearly unity, and the value of the integral depends only on the lower limit of integration.

8. *The asymptotic forms for the line wings.*—In the extreme wings of the line (where the asymmetric distribution eventually dominates, except if $p=2$), the calculation of $J_s(b)$ is simplified and extended by the use of the following asymptotic relations:

Consider equation (62), namely,

$$J_s(b)=A\int_{\mu}^{\infty}\frac{\exp\left[-y\,(\mu/\beta)^{3/p}\right]d\beta}{\left[(b-\beta)^2+\beta^{2/p}\right]\beta^{1+2/p}}. \tag{62}$$

Now, for very large positive values of b, that is, at great distances from the line center, only the regions in which β is comparable in magnitude to b will contribute appreciably to the integral, for the integrand can differ from zero only when $\beta \simeq b$. Further, when β is very large, the exponential term approaches unity. Accordingly, the integral may be replaced by

$$\frac{1}{b^{(p+2)/p}} \int_{-\infty}^{+\infty} \frac{dx}{x^2 + b^{2/p}} = \frac{1}{b^{(p+2)/p}} \times \frac{\pi}{b^{1/p}}; \tag{84}$$

and equation (62) becomes

$$J_s(b) = A \frac{\pi}{b^{(p+3)/p}}. \tag{85}$$

As was anticipated in section 5, this limiting form for J_s in the positive wing is identical in form with that for the usual Holtsmark formula. They both follow the inverse $(p+3)/p$ law; and the difference in the coefficients is due only to the difference in the normalizations—the Holtsmark distribution has itself been normalized to unity, whereas here the total $J(\omega)$ has been so normalized.

It is readily seen from Figure 2 that, in most of the negative-line wing, $J_s(\omega)$ may be represented simply by the function

$$J_s(b) \simeq AR = A \int_{\mu}^{\infty} \frac{\exp[-y(\mu/\beta)^{3/p}]\, \beta^{-(p+2)/p} d\beta}{(b-\beta)^2}. \tag{86}$$

To discuss equation (86) it is convenient to make the change of variable (72), whence we obtain

$$J_s(b) \simeq A \frac{p}{3} \mu^{-2/p} y^{-2/3} \int_0^y \frac{e^{-u} u^{-1/3} du}{(b-Q)^2}, \tag{87}$$

where

$$Q = \mu \left(\frac{y}{u}\right)^{p/3}. \tag{88}$$

Now, when the magnitude of b is very large, Q may be neglected in comparison with it, except in the region of $u \simeq 0$, which contributes very little to the integral anyhow. Hence we may write

$$J_s(b) \simeq A \frac{p}{3} \mu^{-2/p} y^{-2/3} \frac{1}{b^2} \int_0^y e^{-u} u^{-1/3} du, \tag{89}$$

which becomes, for small values of y,

$$J_s(b) \simeq A \mu^{-2/p} \frac{p}{2} \frac{1}{b^2}. \tag{90}$$

Equations (89) and (90) show that in the negative-line wing the decrease of intensity in the asymmetric distribution follows the inverse-square law exactly as does the symmetric distribution. This is because the negative wing is just a superposition of many dispersion-curves.

Table 2 summarizes the principal results of sections 7 and 8, showing how the curve $\mathfrak{J}_s(b)$ ($y \ll 1$) is to be constructed for the different values of b. The curve $\mathfrak{J}_s(b)$ for $p = 4$ is shown in Figure 3. In order to find $J_s(d)$ for any specific line, it is only necessary to find b by means of the relation $b = d\epsilon^{p/(p-1)}$, read off the value of $\mathfrak{J}_s(b)$, and multiply by A. This quantity, when added to the quantity $J_f(d) = c_f(d^2 + 1/\bar{\tau}_f^2)^{-1}$, gives the desired value of J for the point d.

b	$\mathfrak{J}_s = P+Q+R$
$+$ wing…………	$\mathfrak{J} \rightarrow \frac{\pi}{b^{1+p/3}}$
$b > \mu + \frac{3}{2}\mu^{1/p}$………	$\begin{cases} P = \frac{p}{2}\, b^{-2}\, [\mu^{-2/p} - \beta_1^{-2/p}] + \frac{2p}{p-2}\, b^{-3}\, [\beta_1^{1-2/p} - \mu^{1-2/p}] + \ldots .^{*} \\ Q = \frac{p}{4}\, [\beta_1^{-4/p} - \beta_2^{-4/p}] + \frac{p}{2\,(5-p)}\, [\beta_1^{1-5/p} + \beta_2^{1-5/p} - 2\, b^{1-5/p}] - \frac{p}{10}\, b\, [\beta_1^{-5/p} + \beta_2^{-5/p} - 2\, b^{-5/p}]\, ^{\dagger} \\ R = \frac{p}{2p+2}\, \beta_2^{-(2+2/p)} + \frac{2p}{3p+2}\, b\beta_2^{-(3+2/p)} + \frac{3p}{4p+2}\, b^2\beta_2^{-(4+2/p)} + \ldots . \end{cases}$
$\mu < b < \mu + \frac{3}{2}\mu^{1/p}$..	$\begin{cases} P = 0 \\ Q = \frac{p}{4}\, [\mu^{-4/p} - \beta_2^{-4/p}] + \frac{p}{2\,(5-p)}\, [\mu^{1-5/p} + \beta_2^{1-5/p} - 2\, b^{1-5/p}] - \frac{p}{10}\, b\, [\mu^{-5/p} + \beta_2^{-5/p} - 2\, b^{-5/p}]\, ^{\dagger} \\ R = \frac{p}{2p+2}\, \beta_2^{-(2+2/p)} + \frac{2p}{3p+2}\, b\beta_2^{-(3+2/p)} + \frac{3p}{4p+2}\, b^2\beta_2^{-(4+2/p)} + \ldots . \end{cases}$
$-0.5 < b < \mu$……	$\begin{cases} P = 0 \\ Q = \frac{p}{4}\, [\mu^{-4/p} - \beta_2^{-4/p}] + \frac{p}{2\,(5-p)}\, [\beta_2^{1-5/p} - \mu^{1-5/p}] - \frac{p}{10}\, b\, [\beta_2^{-5/p} - \mu^{-5/p}]\, ^{\dagger} \\ R = \frac{p}{2p+2}\, \beta_2^{-(2+2/p)} + \frac{2p}{3p+2}\, b\beta_2^{-(3+2/p)} + \ldots . \end{cases}$
$b < -0.75$………	$\begin{cases} P = 0 \\ Q = 0 \\ R = (-b)^{-(2+2/p)} \int_{\mu/\mu-b}^{1} \left(\frac{1-x}{x}\right)^{1+2/p} dx \end{cases}$
$-$ wing…………	$\mathfrak{J} \rightarrow \frac{p}{2}\, \mu^{-2/p}\, \frac{1}{b^2}$

* $p \neq 3n-1$.

† $p \neq 2+3n$.

From a figure such as Figure 3 it is possible to determine the point in the positiv line wing at which the "statistical distribution" overtakes the damping distribution, fo in the wing we may write $J_f(\omega)$ in the form

$$J_f(d) \simeq \frac{1}{\pi \bar{T} d^2}; \tag{91}$$

or, expressing d in units of $1/\epsilon^{p/(p-1)}$, we find, after some minor transformations,

$$J_f(b) \simeq A \frac{p}{\sqrt{2\pi}} \left(\frac{\pi}{2}\right)^{1/(p-1)} \frac{1}{b^2}. \tag{92}$$

Now the curve

$$J_f(b) = \frac{4}{\sqrt{2\pi}} \left(\frac{\pi}{2}\right)^{1/3} \frac{1}{b^2}, \tag{93}$$

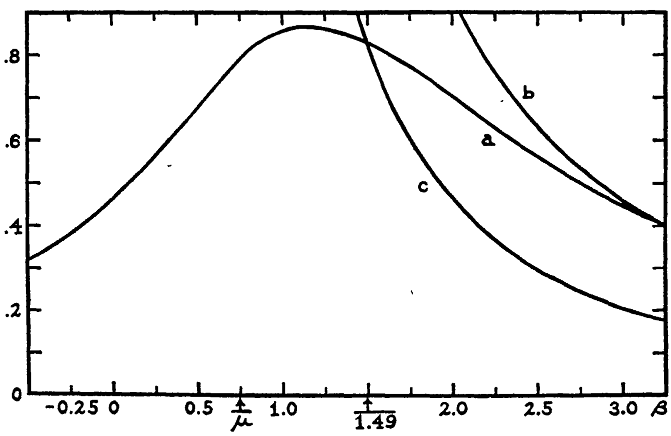

FIG. 3.—*a*, The curve $\mathfrak{J}_s(b)$ for $p = 4$. *b*, The asymptotic curve $\pi/b^{(p+3)/3}$ (eq. [85]), which holds fo large values of b. *c*, The wing of the damping distribution (eq. [93]).

representing the wings of the damping curve, is plotted in Figure 3. Thus we find tha for $p = 4$ the asymmetric curve overtakes the damping curve at the distance

$$d_g = \frac{1.49}{\epsilon^{4/3}}. \tag{94}$$

9. *Concluding remarks on the application of the Burkhardt theory.*—Before the fore going procedure can be applied to the calculation of actual lines, there are three factor to be considered. In the first place, it will be recalled that the effect of collisions takin place beyond the Weisskopf radius ρ_0 has been neglected. The net effect of such "weak' collisions is to produce a slight shift of the damping portion of the line.[20] Consistent wit the adoption of the Weisskopf radius as the dividing line between "effective" and "weak' collisions, the magnitude of the line shift is given as the average frequency displacemen during the "weak" collisions. Thus,

$$\bar{\Delta} = \frac{\int_0^{\Delta_0} \Delta \tau_s(\Delta) W(\Delta) \, d\Delta}{\int_0^{\Delta_0} \tau_s(\Delta) W(\Delta) \, d\Delta}; \qquad \Delta = \gamma \frac{C_p}{\rho^p}; \tag{95}$$

[20] For a detailed discussion of the origin and amount of this shift, see Unsöld, *Vierteljahrschr. d Deutsch. Astr. Gesellsch.*, **78,** 213, 1943, or Lindholm, *Über die Verbreiterung und Verschiebung vo Spektrallinien.* The latter author has made a careful investigation of this problem.

and, upon carrying out the necessary substitutions and integrations, we find, in particular, that, if $p = 4$ and $y \ll 1$,

$$\bar{\Delta}_4 = \frac{1.86}{\bar{\tau}} . \tag{96}$$

While this value of the line shift must undoubtedly be of the right order of magnitude, it is not felt that its exact value should be taken too seriously, because of the rather perfunctory manner in which it has been obtained.

In calculating the present intensity distribution, it has been tacitly assumed that we are dealing with a single collision, that is, that the density is small enough so that it is sufficient to consider the effect of the nearest neighbor. This assumption can be expressed more precisely by the statement that the fundamental distance considered in this problem, namely, the collision radius ρ_0, should be small compared to the mean distance between the radiating atom and the perturbing particles, which we may call r_0. When the density of the perturbing particles is much greater than that of the perturbed atoms, we have, simply,

$$r_0 = 0.554 N^{-1/3} . \tag{97}$$

Thus we may regard the quantity ρ_0/r_0 as a measure of the relative importance of the effect of multiple collisions. It is easily seen from equations (24) and (97) that the dependence of ρ_0/r_0 upon the physical conditions is given by

$$\frac{\rho_0}{r_0} \propto \frac{N^{1/3} C^{1/(p-1)}}{\bar{v}^{1/(p-1)}} . \tag{98}$$

It is interesting to note that ρ_0/r_0 depends upon the pressure and temperature in exactly the same ratio as does $\bar{\tau}_s/\bar{\tau}$ (cf. eq. [50]), except that in the present case the variation is as the cube root and is therefore much slower. As a practical upper limit for the validity of the single-collision approximation we shall adopt the value $\rho_0/r_0 \sim \frac{1}{2}$, which was suggested by Unsöld[5] and does not seem too large, as the effect of multiple collisions must fall off exponentially.

There is one final question to be settled, namely, the method of determining the proper constant of interaction to be used for any given line. The physical problem of calculating this quantity necessarily has a different solution for each value of the orientation of the perturbed atom with respect to the perturber, that is, for all values of the magnetic quantum number m. Therefore, to calculate the amount of broadening of any line, it is necessary to know whether or not these interactions should be averaged and, if so, in what manner. Thus it must be determined whether the broadening collisions take place adiabatically (preserving the relative orientation of atom and perturber) or nonadiabatically. In the former case we should have to calculate the broadened contour for each orientation of atom and perturber and then average over them, whereas in the latter case we should calculate one broadened contour, using the average value of the interactions.

This question was investigated by Lindholm and is discussed at some length by Unsöld. The principal result is that, roughly speaking, collisions which take place within the Weisskopf radius ρ_0 are adiabatic, while the more distant collisions are nonadiabatic, and the different quantum states will tend to be "shuffled" during any collision. Thus in the region of the statistical theory the adiabatic calculation is the correct one, while, in calculating the proper value of ρ_0 to be used in the damping theory, we are considering regions in which the collisions are not strictly adiabatic, and we must therefore use the average value of the C_p's.[21]

[21] These are the results obtained by considering only the upper state of each line, which is mostly responsible for the size of the perturbation. Also, in performing the averages mentioned, the various interaction energies must, of course, be weighted according to their relative probability of occurrence.

These results have the following application to the present problem of calculating the broadening of the helium lines by protons. The interaction energies for various lines have been calculated in an earlier paper[22] and are listed in Table 2 of that paper. Correct to terms in R^{-4} (which is sufficient for the order of distances which will be considered here), they consist of two terms, the R^{-4} term plus a nonhomogeneity term in R^{-3}. Upon taking for any permitted line the properly weighted average of the interactions for use in calculating ρ_0, it is found that the R^{-3} terms drop out, and thus the collision broadening is determined by the R^{-4} law, with an average C_4.

In the region of the statistical theory we should, strictly speaking, calculate line contours for each orientation separately, a procedure in which both the R^{-3} and the R^{-4} terms would have to be used. However, our primary interest in this connection is in distances less than ρ_0, and it may be shown that for each line considered the interaction energy at the distance ρ_0 is due principally to the R^{-4} term. Hence, as a first approximation, the effect of the R^{-3} terms has been neglected in these calculations. A further simplification which has been made is to replace the separate calculations for each orientation by one calculation, using the average of the three-fourths powers of the values of C_4, that is,

$$C_4' = \overline{(C_4^{3/4})}^{\,4/3} . \tag{99}$$

Such a procedure would be correct for the wing of the line, as it may be seen from equations (85), (61), and (58) that in the wing the dependence of $J_s(d)$ on C_p is proportional to $C^{3/p}$. The use of two different values of C_4 in calculating the different parts of the line contour may, quite correctly, be expected to affect the normalization which was made in section 6. In practice, however, it is found that in all but one case (λ 5876) the difference between the straight average of C_4 and the average calculated by means of equation (99) amounts to less than 1.5 per cent, so that the generalized Burkhardt theory has been used with no change.

III. APPLICATION TO THE HELIUM LINES IN τ SCORPII

10. *Calculation of the stellar-line contours.*—Using the methods of Section II, we have calculated the effect of proton interactions on the broadening of helium lines for the physical conditions existing in the atmosphere of the B0 star, τ Scorpii.[23] The necessary physical data for the calculation of the atomic absorption coefficients (considering only the pressure broadening) are the density of protons, $\log N_P = 14.48$, from which we derive $r_0 = 8.25 \times 10^{-6}$ cm, and the temperature, $T = 28{,}150°$, from which we find $\log \bar{v} = 6.4330$. Knowing these quantities, together with the proper values of the interactions which were discussed in section 9, we are able to calculate the intensity distributions $J(\nu)$.

The calculation of the form of the stellar absorption lines was made by using Unsöld's semiempirical relation

$$\frac{1}{R} = \frac{1}{R_c} + \frac{1}{\kappa_\nu}, \tag{100}$$

in which R_c denotes the observed central "intensity," and κ_ν is the line-absorption coefficient at the frequency ν; thus

$$\kappa_\nu = \frac{2\pi^2 e^2}{m c} N H f J(\nu) . \tag{101}$$

[22] Krogdahl, *Ap. J.*, **102**, 64, 1945.

[23] A. Unsöld, *Zs. f. Ap.*, **21**, 22, 1941.

It is well established that, with the possible exception of the very center of the line, this formula gives fairly accurate results over a large range of cases.[24] Further, it was felt that, in view of the present approximations to the calculation of $J(\nu)$, a more rigorous discussion of the manner in which the lines are formed is not justified at this stage. For the same reason the effects of natural line damping and Doppler broadening have been neglected. That the natural damping is very small compared to the collisional damping may be seen from the fifth column of Table 3. The neglect of Doppler effect could be serious only for weak lines.

Line contours have been calculated in this manner for eight of the more interesting helium lines observed in τ Scorpii. Table 3 lists them and contains the necessary observed and derived quantities pertaining to each. It is at once apparent from the table that we are dealing here with much larger collision cross-sections and damping constants than have heretofore been encountered in astrophysical problems. The values of ρ_0, for example, are of the order of a hundred times the first Bohr radius, and the quantities in the fifth column of Table 3 should be compared with $1.24 \times 10^{-22}\nu_0^2$, the half half-width for natural line damping (thus, 5.56×10^7 for λ 4471!).

TABLE 3

λ	Line	$\rho_0 \times 10^6$	$\bar{\tau} \times 10^{11}$	$\frac{1}{\bar{\tau}_f} \times 10^{-10}$	$\bar{\tau}_s \times 10^{12}$	$\frac{\bar{\tau}_s}{\bar{\tau}_f}$	$\frac{\rho_0}{r_0}$	y	$\bar{\Delta}(A^0)$	$d_g(A^0)$	log NH	f	R_c
5876..	2p−3d	0.47	177.	.....	.....	......	0.06	......	....	>8.	14.95	0.553	0.33
4471..	2p−4d	2.92	4.56	2.27	1.55	0.035	.35	0.03	−0.04	0.95	14.95	.129	.48 or .30
4026..	2p−5d	5.40	1.33	9.54	2.86	0.27	.65	.20	− .12	0.42	14.95	.0512	.42
4713..	2p−4s	0.93	45.09	0.22	0.49	0.001	.11	.001	− .005	3.3	14.95	.0052	(.16)
4121..	2p−5s	1.59	15.29	0.66	0.84	0.006	.19	.005	− .011	1.5	14.95	.0001	.26
4921..	2P−4D	3.32	3.53	2.99	1.76	0.053	.40	.05	− .07	1.0	14.42	.118	(.16)
4388..	2P−5D	7.61	0.67	37.2	4.03	1.50	.92	.56	− .28	0.35	14.42	.0416	.26
3965..	2S−4P	2.07	9.04	1.12	1.09	0.012	0.25	0.01	+0.017	1.1	14.06	0.057	(0.24)

It will be seen that for most of the lines the ratio $\bar{\tau}_s/\bar{\tau}_f$ is quite small, meaning that for these lines the collision damping accounts for by far the greater portion of the atomic absorption coefficient. It does not necessarily follow from this, however, that such lines will show no asymmetry in the stellar spectrum, for, no matter how small a part the asymmetric distribution plays, if we consider distances far out in the positive wing, it will eventually overtake the damping curve, in fact, at the distance d_g (eq. [94]). Now it is known from the theory of line formation that beyond a certain stage the addition of more absorbing atoms fills out the wings of an absorption line, so that, no matter how small the ratio $\bar{\tau}_s/\bar{\tau}_f$ might be for a given line, if it were formed by enough absorbing atoms the asymmetry would eventually show up. This is precisely the situation encountered here, as some of the lines are fairly strong. For this reason the quantity d_g, in the eleventh column, gives a better indication than does $\bar{\tau}_s/\bar{\tau}_f$ as to how much asymmetry will be present in a stellar line.

As was anticipated in section 9, the variation of ρ_0/r_0 for these lines is very similar to that of $\bar{\tau}_s/\bar{\tau}_f$. In fact, it appears that, as soon as $\bar{\tau}_s/\bar{\tau}_f$ becomes of the order of 0.1, we are also entering the region where the nearest-neighbor approximation may no longer be strictly correct.

The quantities log NH and R_c, used in computing the stellar lines, are taken from Unsöld's work. The f-values, with the exception of those for λ 4713 and λ 4121 (Hylleraas), are due to Goldberg.

[24] See, e.g., Unsöld, *Physik der Sternatmosphären*.

11. *Description of the individual lines.*—We shall now discuss the general appearance of the stellar lines. Because of the approximations made in regard to both the theory of the pressure broadening and the mechanism of line formation, it is not to be expected that the line contours given here are necessarily final. It seems clear, however, that the general features found by this method must be real.

In the line contours illustrated in Figures 4–8 the shifts $\overline{\Delta}$ have been neglected, as their effect on the equivalent widths (sec. 12) is extremely small. This is justified as long as $\overline{\Delta} \ll d_g$.

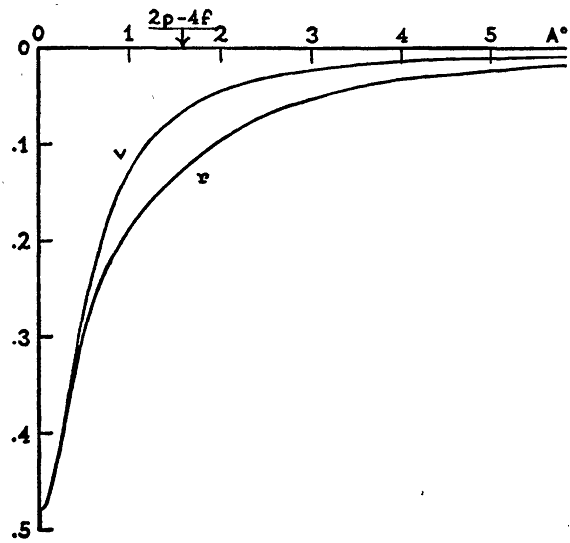

FIG. 4.—λ 4471, with R_c = 48 per cent. The letters *r* and *v* denote the red and violet wings, respectively. The arrow indicates the zero-field position of the $2p$–$4f$ line, which lies, of course, to the violet of λ 4471.

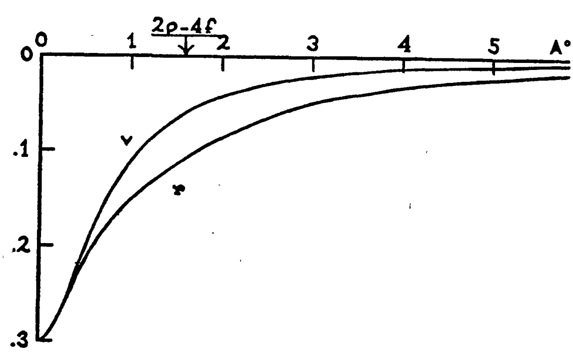

FIG. 5.—λ 4471, with R_c = 30 per cent

λ *4471.*—Unsöld lists two rather different values for the central depth of this line, namely, 48 and 30 per cent. Contours have been calculated for each case and are shown in Figures 4 and 5. For the sake of better comparison, both the red and the violet wings are plotted on the same side of the center. We see that, because of the great strength of the line, it does show asymmetry, despite the fact that $\overline{\tau}_s/\overline{\tau}_f$ is very small.

Perhaps the most interesting feature of the line, however, is the comparative strength and extent of the violet wing, which is due almost entirely to the collisional broadening. Thus, at the zero-field position of the 2p–4f line, which is indicated on the diagrams, the residual intensity in the 2p–4d line amounts to as much as 6 per cent and only drops below 1 per cent at a distance of 4.5 A from the line center. That this result is practically independent of R_c is seen from a comparison of Figures 4 and 5.

λ *4921*.—Because the constants of interaction for this line (2P–4D) are practically the same as for λ 4471, the features of the line are qualitatively the same, except that the whole line is weaker. The residual intensity in the violet wing at the 2P–4F position is 3 per cent, and it drops to 1 per cent at 3.2 A from the center.

λ *4026*.—This is the strongest helium line observed in τ Scorpii; and, while the value of τ_0/r_0 is as much as 0.65, it is not felt that this is too much greater than the arbitrarily chosen limit 0.5 but that the present methods may be applied with profit. It is clear, however, that the possible effect of multiple collisions should be considered.

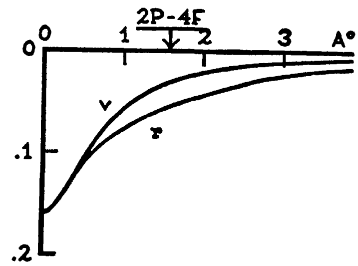

FIG. 6.—λ 4921, R_c = 16 per cent

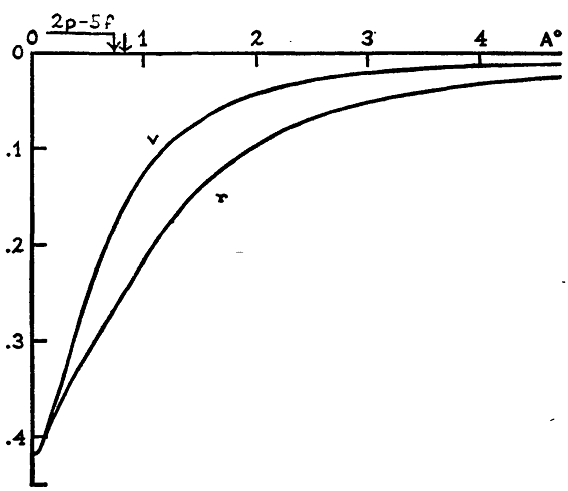

FIG. 7.—λ 4026, R_c = 42 per cent. The unmarked arrow indicates the approximate position of forbidden $2p$–$5g$.

For this line $\bar{\tau}_s/\bar{\tau}_f$ is of an appreciable size, namely, 0.27; and, because the line is also very strong, there are two reasons why its asymmetry is very much more marked than that of, say, λ 4471. The violet wing, which, however, represents the effect of the collisional broadening, has a residual intensity of nearly 18 per cent at the 2p–5f position, about 16 per cent at 2p–5g, and 1 per cent at 5.1 A from the line center, as compared with a central intensity of 42 per cent.

The profile given here is, at best, a first approximation, to the extent that we have assumed that $y = 0.20$ is negligible compared to unity. Also we have neglected $\overline{\Delta}$, which in this case amounts to two-sevenths of d_g.

λ *4388*.—Here, if the present methods were applicable, we should have an example of a line in which the asymmetric curve is dominant, as is indicated by the ratio $\bar{\tau}_s/\bar{\tau}_f$ = 1.50. At the same time, however, the value of ρ_0 is practically the same as that of r_0, which means that the use of the single-collision approximation is almost certainly no longer justifiable.

λ *3965*.—The principal interesting feature of this line is that the asymmetry is to the violet. However, it hardly shows up in the stellar line, as can be seen from the ratio $\bar{\tau}_s/\bar{\tau}_f = 0.012$ and the fact that the line is comparatively weak.

λ *4713*.—This line has an extremely small collision cross-section and degree of asymmetry, as should be expected from the fact that its upper level is a low *s*-level. Its calculated contour for τ Scorpii shows no asymmetry.

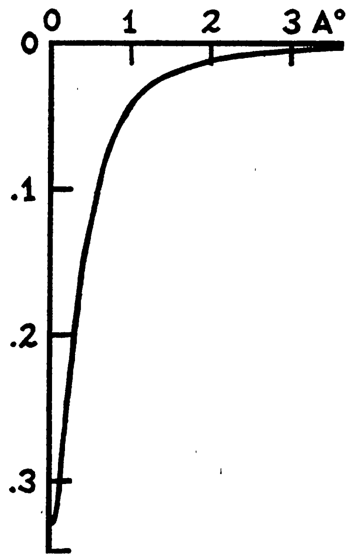

FIG. 8.—λ 5876, R_c = 33 per cent. The contour is symmetrical

TABLE 4

λ	Line	Equivalent Width (A) (Calculated)	Equivalent Width (A) (Observed)	R_c	Remarks
5876	2p–3d	0.402	0.562	0.33	
4471	4p–4d	{1.175	0.955	.48	
		{0.955	0.955	.30	
4026	2p–5d	1.135	1.259	.42	ρ_0/r_0=0.65
4713	2p–4s	0.035	0.229	(.16)	R_c uncertain
4121	2p–5s	0.008	0.288	.26	Blend; $f=10^{-4}$
4921	2P–4D	0.484	0.831	(.16)	R_c questionable
4388	2P–5D	(0.690)	0.661	.26	ρ_0/r_0=0.92
3965	2S–4P	0.091	0.129	(0.24)	Blend; R_c uncertain

λ *4121*.—In this case also, ρ_0 and $\bar{\tau}_s/\bar{\tau}_f$ are quite small, and the resultant contour shows no asymmetry. However, the *f*-value given by Hylleraas seems incredibly small;[25] and, if it should be in error, it is possible that the asymmetry would show up.

λ *5876*.—This line is of especial interest, as it is the only one of the lines considered which is the leading member of a series. Since, for the upper state, we have $n = 3$, the interaction and therefore the collision cross-section and ratio $\bar{\tau}_s/\bar{\tau}_f$ are very small (compared to other lines of the series). As the line is not especially strong in τ Scorpii, it therefore shows no asymmetry. The shift, if any, is to the violet.

[25] Cf. Unsöld's remarks on the subject (*Zs. f. Ap.*, **21**, 22, 1941).

12. *The equivalent widths of the lines.*—Perhaps the most significant result of these calculations, however, is that contained in Table 4, of which the third column gives the equivalent widths of the eight lines as determined by planimeter measurement. The two values listed for λ 4471 correspond to the two empirical values for R_c. The width given for λ 4388 is simply an estimate based on the behavior of the wings of the line and should not be taken seriously.

The fourth column contains the observed equivalent widths adopted by Unsöld from several sets of measurements. A comparison of the two columns leaves little doubt that for most of the lines the agreement is good. (To be sure, the values for λ 4713 and λ 4121 are too low—but, as has been mentioned before, the f-value for the latter may be questionable.) Further, the agreement between calculated and observed intensity holds over a considerable range of line strengths (at least a factor of 10), and for several distinct types of line series. Now the significance of these results for the theory of line broadening by direct proton interactions lies in the fact that they are consistent with the observed physical parameters of the atmosphere. In other words, they suggest that the direct interactions may be entirely capable of accounting for the observed strengths of the helium lines, at the temperature and electron density to be expected from other considerations (such as the number of observable H lives, Stark effect of same, etc.).

To summarize the conclusions of sections 10–12, a preliminary investigation of the effect of direct proton interactions on the broadening of the helium lines in τ Scorpii leads to two results. The first is that the physical conditions in the atmosphere are such that, for most lines, collisional damping of a modified Lorentz-Weisskopf type is primarily responsible for the broadening. Secondly, it is found that broadening of this type can furnish a satisfactory explanation of the equivalent widths of the lines, no qualifying assumptions being necessary. Thus the answer to the problem proposed in the introductory section is that a consideration of the effect of direct interactions on line broadening shows it to be a most fruitful method of attack, which should be discussed, if possible, by more refined methods.

I wish to express my sincere thanks to Dr. S. Chandrasekhar for reading the manuscript and for the numerous invaluable discussions of many aspects of the problem. The present work was started at the Yerkes observatory and completed at Evanston, Illinois.