Chapter 3

Tensors in Rectilinear Coordinates in Three Dimensions

1. Reversible Linear Transformations

Much of the apparatus of vector and tensor analysis can be exhibited in its application to two dimensions. The definition of tensors, their algebra, and the differentiation of tensors are not essentially different in three or more dimensions; these results may therefore for the most part be taken over in a routine and straightforward way. However, an additional degree of freedom offers new possibilities for generalization as well as application. It is these which we propose to explore, first in rectilinear coordinates, then in generalized coordinates.

A rectilinear coordinate system is defined by any three non-coplanar straight lines which intersect at a common point, the origin. The lines are the coordinate axes, along which scales of length $s_{(i)}$ are given. The contravariant rectilinear coordinates x^i of any point $\mathrm{P}(x^i)$ may be found by a simple construction which is merely the straightforward generalization of the corresponding construction in two dimensions; this is detailed in Appendix 3.1. It is the “parallel” projection. We may at the same time define covariant rectilinear coordinates by “perpendicular” projection; this too is described in Appendix 3.1.

The relation between the covariant and contravariant rectilinear coordinates with respect to a given set of axes is, as in two dimensions,

(1.1) $$x_i = g_{ij} x^i \text{ and } (i, j = 1, 2, 3) \text{ with}$$

(1.2) $$g_{ij} g^{jk} = \delta_i^k .$$

The covariant fundamental tensor g_{ij}, and the contravariant fundamental tensor g^{ij} are expressible in terms of the angles between the coordinate axes and the coordinate planes as

(1.3) $$g_{ij} = \begin{Vmatrix} s_{(1)}^2 & s_{(1)} s_{(2)} \cos\theta_{12} & s_{(1)} s_{(3)} \cos\theta_{13} \\ s_{(1)} s_{(2)} \cos\theta_{12} & s_{(2)}^2 & s_{(2)} s_{(3)} \cos\theta_{23} \\ s_{(1)} s_{(3)} \cos\theta_{13} & s_{(2)} s_{(3)} \cos\theta_{23} & s_{(3)}^2 \end{Vmatrix}$$

and

(1.4)

$$g^{ij} = \frac{\sin\theta_{23}\sin\theta_{13}\sin\theta_{12}}{s_{(1)}s_{(2)}s_{(3)}g}\left\| \begin{matrix} \dfrac{\sin\theta_{23}}{s_{(1)}} & \dfrac{-\sin\theta_{13}\cos\alpha_{132}}{s_{(2)}} & \dfrac{-\sin\theta_{12}\cos\alpha_{123}}{s_{(3)}} \\ \dfrac{-\sin\theta_{23}\cos\alpha_{123}}{s_{(1)}} & \dfrac{\sin\theta_{13}}{s_{(2)}} & \dfrac{-\sin\theta_{12}\cos\alpha_{213}}{s_{(3)}} \\ \dfrac{-\sin\theta_{23}\cos\alpha_{123}}{s_{(1)}} & \dfrac{-\sin\theta_{13}\cos\alpha_{213}}{s_{(2)}} & \dfrac{\sin\theta_{12}}{s_{(3)}} \end{matrix} \right\|.$$

Here g is the determinant which has the common value of

(1.5)

$$g\left[s_{(1)}s_{(2)}s_{(3)}\right]^{-2} = \left[\sin\theta_{13}\sin\theta_{23}\sin\alpha_{132}\right]^2$$

$$= \left[\sin\theta_{12}\sin\theta_{13}\sin\alpha_{213}\right]^2 = \left[\sin\theta_{23}\sin\theta_{12}\sin\alpha_{123}\right]^2$$

$$= \left[1 + 2\cos\theta_{12}\cos\theta_{13}\cos\theta_{23} - \cos^2\theta_{12} - \cos^2\theta_{13} - \cos^2\theta_{23}\right] \neq 0\,.$$

The angles α_{ijk} are the dihedral angles along the OX^j – axes between the respective planes containing the axes $\left(OX^i, OX^j\right)$ and $\left(OX^j, OX^k\right)$ (see Fig. 61.)

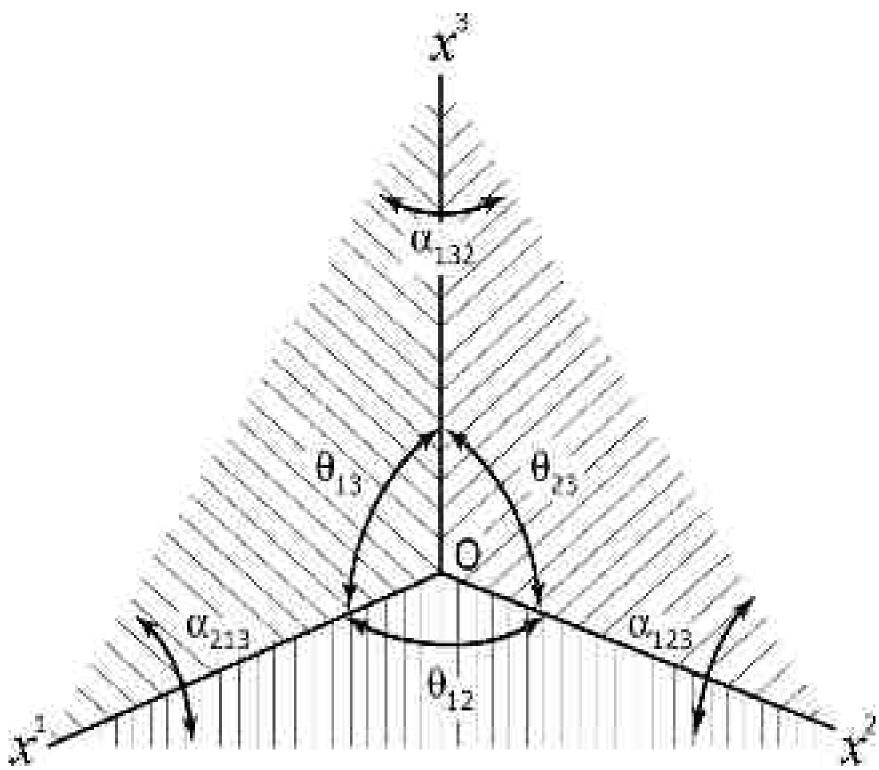


Figure 61

We may now consider any two rectilinear coordinate systems with a common origin O. The coordinates of a point P may be given in either system. There must therefore be a relation between these two sets of coordinates. It is, predictably, a reversible linear homogeneous transformation

$$\bar{x}^i = \bar{a}_j^i x^j, \quad (i, j = 1, 2, 3),$$

where the coefficients $\bar{a}_j^i$ are, aside from scale factors, known functions of the angles between the three axes (Appendix 3.1, equations (3.1.16)). Conversely, any reversible linear homogeneous transformation of rectilinear coordinates may be shown to imply a certain change of axes and possibly changes of scale (Appendix 3.2).

We shall henceforth consider these points established and not labor them further. Though the extension to a third dimension has been at the price of a more tedious geometrical analysis and lengthier formulae, the general argument runs just as it did in two dimensions. It is to be understood, of course, that in three dimensions indices have the range 1 to 3. We note, too, that the meaning of the inner product is exactly the same in three dimensions as in two.

Ex. (1.1) (a) Given that

$$\theta_{12} = 125° \, 40' \, 14'', \; \theta_{13} = 53° \, 56' \, 12', \textit{ and } \theta_{23} = 98° \, 51' \, 16'',$$

find the covariant fundamental tensor in this coordinate system if all the scale factors are unity. (b) Calculate g^{ij} as the inverse of g_{ij}. (c) Check your answers by using equation (1.4).

Ans. (*a*)
$$g_{ij} = \begin{Vmatrix} 1.0 & -0.58312 & 0.58868 \\ -0.58312 & 1.0 & -0.15392 \\ 0.58868 & -0.15392 & 1.0 \end{Vmatrix};$$

(*b*)
$$g^{ij} = \begin{Vmatrix} 2.4691 & 1.2456 & -1.2618 \\ 1.2456 & 1.6526 & -0.4789 \\ -1.2618 & -0.4789 & 1.6691 \end{Vmatrix}$$

(*c*) (*Hint:* $\alpha_{213} = 73° \, 14' \, 43''$, $\alpha_{123} = 51° \, 34' \, 15''$, $\alpha_{132} = 128° \, 4' \, 8''$.)

Ex. (1.2) Given that the rectilinear contravariant coordinates $\bar{x}^i$ are related to the Cartesian coordinates x^i by the transformation

$$x^1 = 3\bar{x}^1 - \bar{x}^2 + 2\bar{x}^3, \; x^2 = \bar{x}^1 + 4\bar{x}^2 + 3\bar{x}^3,$$
$$x^3 = -2\bar{x}^1 - 3\bar{x}^2 + 5\bar{x}^3,$$

what are the components of the fundamental tensor in the new coordinate system? (Hint: $g_{ij} = \delta_{ij}$).

Ans.
$$\bar{g}_{ij} = \begin{Vmatrix} 14 & 7 & -1 \\ 7 & 26 & -5 \\ -1 & -5 & 38 \end{Vmatrix}.$$

Ex. (1.3) Determine the inner product of $a^i = (1, 3, -2)$ *and* $b_i = (2, -4, 1)$.

Ans. $a^i b_i = 1 \cdot 2 + 3 \cdot (-4) + (-2) \cdot 1 = -12$.

Ex. (1.4) What is the magnitude of the vector whose components in a Cartesian coordinate system are $\mathrm{v}^i = (3, -1, 8)$?

Ans. $|\mathrm{v}^i| = (3^2 + [-1]^2 + 8^2)^{1/2} = \sqrt{74}$.

Ex. (1.5) Show that we must always have

$$(g_{ij})^2 < g_{ii} g_{jj}, \quad i \neq j, \quad (\textit{no summation}).$$

(Hint: use equation (1.3).)

2. Covariant Vectors, Contravariant Vectors, and Duality

A vector in three dimensions, like a vector in two dimensions, is a set of components which transform in identically the same way as the rectilinear coordinates. This implies that a contravariant vector transforms as

$$\bar{\mathrm{v}}^i = \frac{\partial \bar{x}^i}{\partial x^j} \mathrm{v}^j$$

and a covariant vector transforms as

$$\bar{\mathrm{v}}_i = \frac{\partial x^j}{\partial \bar{x}^i} \mathrm{v}_j$$

when one goes from coordinates x^i to a new system $\bar{x}^i$.

However, certain distinctions between covariant and contravariant vectors may be brought out more clearly in three dimensions than in two. For example, consider the vector

$$u^i = (u^1, 0, 0).$$

It is parallel to the x^1 – axis since its "parallel" projections upon the other two axes are both zero. Hence all vectors parallel to the x^1 – axis are of the form αu^i, where α is an invariant.

Now consider a position vector

$$p^i = (p^1, p^2, 0).$$

It represents a vector in the (x^1, x^2) – plane. Hence all vectors in the (x^1, x^2) – plane are proportional to some p^i. We say that the collection of all such vectors ***spans*** the (x^1, x^2) – plane. This plane may also be identified as the two-dimensional sub-space $x^3 = 0$. More generally, then, in rectilinear coordinates the sub-space $x^3 = \alpha^3 = \text{constant}$ is spanned by all vectors of the form $p^i = (x^1, x^2, \alpha^3)$.

Consider next a covariant vector

$$n_i = (0, 0, n_3).$$

Since
$$n_i p^i = n_3 \alpha^3 ,$$
it is evident that n_i is orthogonal to the (x^1, x^2)-plane $\alpha^3 = 0$, hence to *both* the x^1- and x^2-axes. Clearly, all vectors orthogonal to the (x^1, x^2)-plane are proportional to n_i. The vectors $p^i = (p^1, p^2, 0)$ and $n_i = (0, 0, n_3)$ are said to span **mutually dual spaces**. Similarly, $(p^1, 0, 0)$ and $(0, n_2, n_3)$ span mutually dual spaces as do $(0, n_2, 0)$ and $(p^1, 0, p^3)$.

It is of interest to note that the contravariant form of n_i is
$$n^i = g^{ij} n_j = (g^{13} n_3, g^{23} n_3, g^{33} n_3) = n_3 (g^{13}, g^{23}, g^{33}) .$$
From this and the two analogous equations we see that the components of the contravariant fundamental tensor are proportional to the contravariant components of the normals to the coordinate planes. We thus have an additional geometric interpretation of the contravariant fundamental tensor.

Ex. (2.1) Show that the g_{ij} are proportional to the covariant components of the vectors parallel to the respective coordinate axes.

We thereby see that the space spanned by $(p^1, p^2, 0)$ is dual to the space spanned by $(0, 0, n_3)$, etc. By an entirely similar argument, we can show that the space spanned by $(p^1, 0, 0)$ (the x^1-axis) is dual to the space spanned by $(0, n_2, n_3)$. This latter is evidently the plane common to the normals of the (x^1, x^2)- and (x^1, x^3)-plane, hence the plane normal to the x^1-axis. (Verify by taking inner products.)

We may extend the notion of duality of spaces to duality of tensors by introducing the completely antisymmetric third order tensor whose Cartesian components are

$$(2.1) \qquad e_{ijk} = \begin{cases} +1 \text{ if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1 \text{ if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0 \text{ otherwise} . \end{cases}$$

It may alternatively be defined in Cartesian coordinates as
$$e_{ijk} = \delta^{123}_{ijk} ,$$
where δ^{lmn}_{ijk} is the **generalized Kronecker delta** in three dimensions, conveniently expressible as

$$(2.2) \qquad \delta^{lmn}_{ijk} = \begin{vmatrix} \delta^l_i & \delta^l_j & \delta^l_k \\ \delta^m_i & \delta^m_j & \delta^m_k \\ \delta^n_i & \delta^n_j & \delta^n_k \end{vmatrix} .$$

Evidently this tensor will have components in other coordinate systems which are

$$\boldsymbol{\varepsilon}_{pqr} = \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial x^k}{\partial \bar{x}^r} e_{ijk}.$$

Clearly this is antisymmetric in any two free indices, such as p and q, for

$$\boldsymbol{\varepsilon}_{qpr} = \frac{\partial x^i}{\partial \bar{x}^q}\frac{\partial x^j}{\partial \bar{x}^p}\frac{\partial x^k}{\partial \bar{x}^r} e_{ijk} = \frac{\partial x^j}{\partial \bar{x}^p}\frac{\partial x^i}{\partial \bar{x}^q}\frac{\partial x^k}{\partial \bar{x}^r} e_{ijk} = -\frac{\partial x^j}{\partial \bar{x}^p}\frac{\partial x^i}{\partial \bar{x}^q}\frac{\partial x^k}{\partial \bar{x}^r} e_{jik}$$

$$= -\frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial x^k}{\partial \bar{x}^r} e_{ijk} = -\boldsymbol{\varepsilon}_{pqr}.$$

Since it is completely antisymmetric, its non-zero components can only be $\pm\boldsymbol{\varepsilon}_{123}$. This, however, is

$$\boldsymbol{\varepsilon}_{123} = \frac{\partial x^i}{\partial \bar{x}^1}\frac{\partial x^j}{\partial \bar{x}^2}\frac{\partial x^k}{\partial \bar{x}^3} e_{ijk}.$$

Written out in full, the non-vanishing terms are

$$\boldsymbol{\varepsilon}_{123} = \frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^2}\frac{\partial x^3}{\partial \bar{x}^3} + \frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^3}{\bar{x}^2}\frac{\partial x^1}{\partial \bar{x}^3} + \frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^2}\frac{\partial x^2}{\partial \bar{x}^3}$$

$$- \frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^2}\frac{\partial x^3}{\partial \bar{x}^3} - \frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^2}\frac{\partial x^1}{\partial \bar{x}^3} - \frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^3}{\partial \bar{x}^2}\frac{\partial x^2}{\partial \bar{x}^3} = \left|\frac{\partial x^m}{\partial \bar{x}^n}\right|.$$

Therefore, in general, we must have that

$$\boldsymbol{\varepsilon}_{pqr} = \left|\frac{\partial x^m}{\partial \bar{x}^n}\right| e_{pqr}. \tag{2.3}$$

As in two dimensions, we can easily show that $\left|\frac{\partial x^m}{\partial \bar{x}^n}\right| = \pm\sqrt{g}$, where g is the determinant of the fundamental tensor. The plus sign is to be taken with right-handed coordinate systems, the minus sign with left-handed ones. (See Exercises (2.2, 2.3).)

Ex. (2.2) The transformation

$$x^1 = -\bar{x}^1,\ x^2 = \bar{x}^2,\ x^3 = \bar{x}^3$$

converts a right-handed Cartesian coordinate system x^i *into a left-handed system* $\bar{x}^i$ *and vice versa. (a) Show that*

$$\left|\frac{\partial x^m}{\partial \bar{x}^n}\right| = -1.$$

(b) Draw the axes for the two systems and indicate their positive directions.

Ex. (2.3) (a) Does the transformation

$$x^1 = -\bar{x}^1,\ x^2 = \bar{x}^2,\ x^3 = -\bar{x}^3,$$

where the x^i is a right-handed Cartesian system, yield a right-handed or a left-handed system? (b) What test discriminates between the two?

Ans. (a) Right-handed ; (b) right-handed if $\left|\dfrac{\partial x^m}{\partial \bar{x}^n}\right| = > 0$.

Ex. (2.4) Show that $\boldsymbol{\varepsilon}^{pqr} = \left|\dfrac{\partial \bar{x}^n}{\partial x^m}\right| e^{pqr} = \dfrac{e^{pqr}}{\sqrt{g}}$ *is a completely anti-symmetric tensor, where* $e^{pqr} = \delta^{pqr}_{123}$.

Ex. (2.5) Show that $\boldsymbol{\varepsilon}^{ijk}\boldsymbol{\varepsilon}_{lmn} = \delta^{ijk}_{lmn}$.

With this antisymmetric tensor, $\boldsymbol{\varepsilon}_{ijk}$, we can form products of vectors which are of special interest. Thus, consider the product

$$c_i = \boldsymbol{\varepsilon}_{ijk}\, a^j b^k,$$

called the **vector product** of the vectors a^j and b^k. From the fact that

$$a^i c_i = \boldsymbol{\varepsilon}_{ijk}\, a^i a^j b^k = -\boldsymbol{\varepsilon}_{jik}\, a^j a^i b^k = -\boldsymbol{\varepsilon}_{ijk}\, a^i a^j b^k = -a^i c_i,$$

it is clear that $a^i c_i = 0$, whence a^i and c_i are orthogonal. Similarly, b^i and c_i are orthogonal.

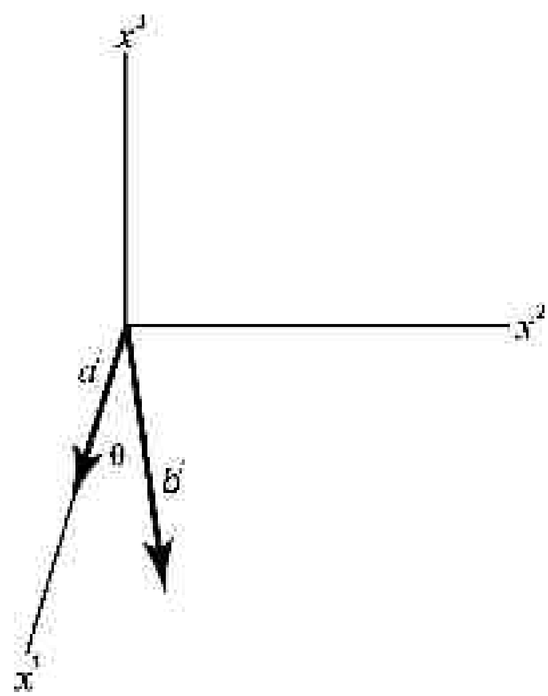


Figure 62

We may interpret the vector product of a^i and b^i very readily by evaluating its components in a conveniently chosen coordinate system. We choose for this purpose a Cartesian system such that the x^1 – axis lies along a^i and the x^2 – axis is perpendicular to it in the plane of a^i and b^i (see Fig. 62). Then $a^i = (a, 0, 0)$, $b^i = (b\cos\theta, b\sin\theta, 0)$ and

$$c_i = (0, 0, ab\sin\theta). \tag{2.4}$$

From this we see that the vector product of two vectors is perpendicular to both and has a magnitude equal to the product of their magnitudes times the sine of the angle between them. It is also clear that the vector product is the dual to the space spanned by the two vectors.

Ex. (2.6) In a Cartesian coordinate system, $a^i = (3, -2, 5)$, $b^i = (-4, 1, 6)$. *Find the unit vector perpendicular to* a^i *and* b^i.

Ans. Since $c_i = \varepsilon_{ijk} a^j b^k = (-17, -38, -5)$ *is perpendicular to both* a^j *and* b^k *and since* $|c_i| = \sqrt{1758}$, *the unit vector in the direction of* c_i *has components*

$$\gamma_i = \frac{1}{\sqrt{1758}}(-17, -38, -5).$$

Since this is a Cartesian system, the components of the contravariant unit normal are the same.

Ex. (2.7) What is the angle between the vectors a^i *and* b^i *of Ex. (2.6)?*

Ans. Since $|c_i| = |a^i| \cdot |b^i| \cdot \sin\theta$, *and since* $|a^i| = \sqrt{38}$, $|b^i| = \sqrt{53}$, *we have*

$$\sin\theta = \left(\frac{1758}{38 \cdot 53}\right)^{1/2} = \left(\frac{879}{1007}\right)^{1/2}.$$

To determine the quadrant of θ, *we compute*

$$\cos\theta = \frac{16}{\sqrt{38 \cdot 53}} = \left(\frac{128}{1007}\right)^{1/2}.$$

Hence $\theta = 69° \, 6' \, 47''$.

Ex. (2.8) In a certain oblique coordinate system, the fundamental tensor has components

$$g_{ij} = \left\| \begin{array}{ccc} 1 & -2/3 & 3/2 \\ -2/3 & 4 & 3 \\ 3/2 & 3 & 9 \end{array} \right\|.$$

Find the unit vector perpendicular to $a^i = (1, 4 - 3)$ *and* $b^i = (5, -7, 2)$.

Ans. Since $g = 8$, *we have*

$$c_i = \varepsilon_{ijk} a^j b^k = 2\sqrt{2}(-13, -17, -27)$$

is perpendicular to a^i *and* b^i*. Since the contravariant fundamental tensor is*

$$g^{ij} = \begin{Vmatrix} 27/8 & 21/16 & -1 \\ 21/16 & 27/32 & -1/2 \\ -1 & -1/2 & 4/9 \end{Vmatrix},$$

the magnitude of c_i *must be*

$$\left| c_i \right| = (g^{ij} c_i c_j)^{\frac{1}{2}} = \frac{3}{2}\sqrt{7347}$$

Therefore, the unit vector perpendicular to a^i *and* b^i *has the covariant components*

$$n_i = -\frac{4}{3}\left(\frac{2}{7347}\right)^{1/2} \times (13, 17, 27).$$

Ex. (2.9) Show that the inner product of a vector a^i *with the vector products of vectors* b^i *and* c^i *is an invariant* $\varepsilon_{ijk} a^i b^j c^k$ *equal to* $a\,b\,c \cos\theta \sin\varphi$, *where* a, b, *and* c *are the magnitudes of* a^i, b^i *and* c^i, θ *is the angle between* b^i *and* c^i, *and* φ *is the angle between* a^i *and the plane of* b^i *and* c^i. *(Hint: Choose a Cartesian coordinate system with the* x^1*–axis along* b^i*, the* x^2*–axis in the plane of* b^i *and* c^i*.)*

Ex. (2.10) Determine the vector product of a_i *with that of* b^i *and* c^i.

Ans.

$$\begin{aligned} \varepsilon^{ijk} a_j (\varepsilon_{klm} b^l c^m) &= (\varepsilon^{ijk} \varepsilon_{klm}) a_j b^l c^m \\ &= \delta^{ijk}_{lmk} a_j b^l c^m = (\delta^{ij}_{lm} b^l c^m) a_j \\ &= (b^i c^j) a_j - (b^j c^i) a_j = (a_j c^j) b^i - (a_j b^j) c^i. \end{aligned}$$

More generally, we can say that a tensor

$$\omega_{ij} = \varepsilon_{ijk} \omega^k \tag{2.6}$$

is dual to ω^k, and conversely. As may be inferred from the examples of duality thus far considered, duality implies orthogonality and complementarity both of rank and of covariance-contravariance. The reciprocity of this relationship may be further emphasized by noting that the vector ω^k may be readily recovered from its dual ω_{ij}.

Thus

$$\varepsilon^{lij} \omega_{ij} = \varepsilon^{lij} \varepsilon_{ijk} \omega^k = \frac{1}{\sqrt{g}} e^{lij} \sqrt{g}\, e_{ijk} \omega^k = \delta^{lij}_{ijk} \omega^k = \delta^{lij}_{kij} \omega^k.$$

Now

$$\delta^{lij}_{kij} = \delta^{li1}_{ki1} + \delta^{li2}_{ki2} + \delta^{li3}_{ki3} = \delta^{li}_{ki}$$

since at most only one of the terms of the middle sum can be different from zero. Further,

$$\delta^{li}_{ki} = \delta^{l1}_{k1} + \delta^{l2}_{k2} + \delta^{l3}_{k3} = 2\,\delta^{l}_{k}$$

and l and k must each be different from two of the values of i but the result is in any case zero unless l and k are the same. Hence

$$\frac{1}{2}\,\varepsilon^{lij}\,\omega_{ij} = \frac{1}{2}\,(2\,\delta^{l}_{k}\,\omega^{k}) = \omega^{l}. \tag{2.7}$$

Ex. (2.11) What is the tensor dual to the covariant vector ω_k in three dimensions? (Hint: raise the free indices in equation (2.6) and interchange the summation indices.)

Ans. $\omega^{ij} = \varepsilon^{ijk}\,\omega_k$.

Ex. (2.12) Show that $\delta^{ijk}_{ijk} = 3\,!$ *in three dimensions. (Hint: use the fact that* $\delta^{lij}_{kij} = \delta^{li}_{ki} = 2\,\delta^{l}_{k}$.)

We may illustrate the usefulness of the vector product of vectors by using it to derive some familiar results from spherical trigonometry. Thus consider a unit sphere with center at P as in Fig. 63. Let λ^r, μ^r and ν^r be three non-coplanar radii. They define a spherical triangle with sides a, b and c and angles A, B and C.

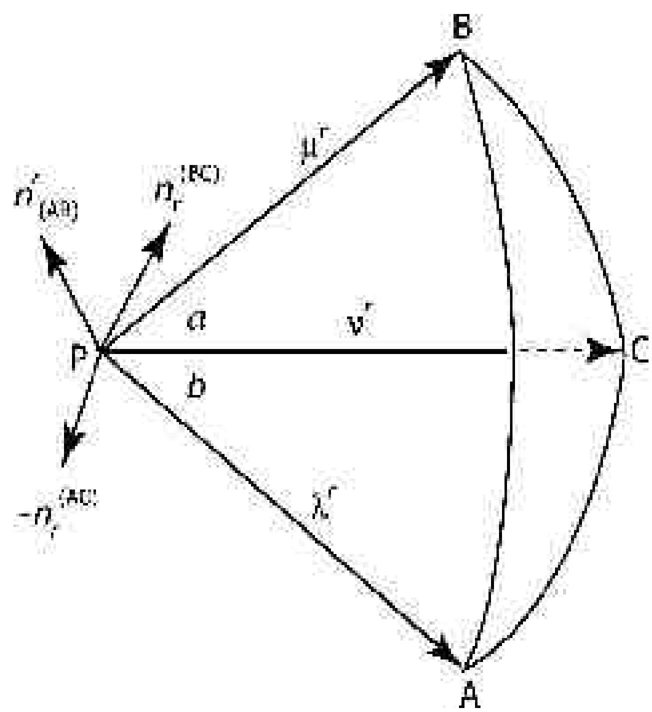


Figure 63

Now consider the invariant

$$\delta^{rst}_{rpq}\,\mu_s\,\lambda_t\,\nu^p\,\mu^q \;=\; \delta^{st}_{pq}\,\mu_s\,\lambda_t\,\nu^p\,\mu^q\,. \tag{2.5}$$

The right-hand side may be written as

$$\begin{aligned}\mu_s\,\lambda_t\,(\nu^s\mu^t - \mu^s\nu^t) &= (\lambda_t\,\mu^t)(\mu_s\,\nu^s) - (\mu_s\,\mu^s)(\lambda_t\,\nu^t)\\ &= \cos c\cos a - \cos b\,,\end{aligned}$$

inasmuch as all three vectors have unit magnitude.

The left-hand side of equation (2.5), however, may be given the form

$$\begin{aligned}\delta^{rst}_{rpq}\,\mu_s\,\lambda_t\,\nu^p\,\mu^q &= (\varepsilon^{rst}\mu_s\,\lambda_t)(\varepsilon_{rpq}\,\nu^p\,\mu^q)\\ &= \Big[\,|\,\mu_s\,|\cdot|\,\lambda_t\,|\,(\sin c\,)\,n^r_{(\mathrm{AB})}\Big]\Big[\,|\,\nu^p\,|\cdot|\,\mu^q\,|\,(\sin a\,)\,n_r^{(\mathrm{BC})}\Big]\\ &= \sin a\sin c\left(n^r_{(\mathrm{AB})}\,n_r^{(\mathrm{BC})}\right),\end{aligned}$$

where $n^r_{(\mathrm{AB})}$ is the unit normal to the plane of λ^r and μ^r and $n_r^{(\mathrm{BC})}$ is the unit normal to the plane of μ^r and ν^r. Therefore the inner product in parenthesis in the preceding expression is

$$n^r_{(\mathrm{AB})}\,n_r^{(\mathrm{BC})} \;=\; 1\cdot 1\cdot\cos(\,180^\circ - \mathrm{B}\,) \;=\; -\cos\mathrm{B}\,.$$

Equating the values thus found for the two sides of the original equation, we then have

$$-\sin a\sin c\cos\mathrm{B} \;=\; \cos a\cos c - \cos b$$

or

$$\cos b \;=\; \cos a\cos c + \sin a\sin c\cos\mathrm{B}\,,$$

the law of cosines of spherical trigonometry.

Next, let us consider the invariant

$$\mathrm{I} \;=\; \varepsilon^{prk}\lambda_p\Big[\varepsilon_{rst}\,\lambda^s\,\nu^t\Big]\Big[\varepsilon_{klm}\,\mu^l\,\lambda^m\Big].$$

Since

$$\varepsilon_{rst}\,\lambda^s\,\nu^t \;=\; (\sin b\,)\,n_r^{(\mathrm{AC})}, \quad \varepsilon_{klm}\,\mu^l\,\lambda^m \;=\; (\sin c\,)\,n_k^{(\mathrm{AB})},$$

we must have $\quad \mathrm{I} \;=\; (\sin b\sin c\,)\,\lambda_p\,\varepsilon^{prk}\,n_r^{(\mathrm{AC})}\,n^{k(\mathrm{AB})}\,,$

where $n_r^{(\mathrm{AC})}$ is the unit normal to the plane of λ^s and ν^t. Now because $\varepsilon^{prk}\,n_r^{(\mathrm{AC})}\,n_k^{(\mathrm{AB})}$ is normal to both normals, it must lie in both the planes of λ^s and ν^t as well as of μ^l and λ^m, hence be along their intersection. That is, we must have

$$\varepsilon^{prk}\,n_r^{(\mathrm{AC})}\,n_k^{(\mathrm{AB})} \;=\; (\sin\mathrm{A}\,)\,\lambda_p\,.$$

The invariant I thus has the value

$$\mathrm{I} \;=\; \sin b\sin c\sin\mathrm{A}\,.$$

On the other hand, we may express I also in the form

$$\mathrm{I} \;=\; \lambda_p\, \varepsilon^{prk} [\, \varepsilon_{rst}\, \lambda^s\, \nu^t\,]\, [\, \varepsilon_{klm}\, \mu^l\, \lambda^m\,]$$

$$\lambda_p\, (\, \varepsilon^{prk}\, \varepsilon_{klm}\,)\, \mu^l\, \lambda^m\, [\, \varepsilon_{rst}\, \lambda^s\, \nu^t\,] \;=\; \lambda_p\, (\, \delta^{prk}_{lmk}\, \mu^l\, \lambda^m\,)\, \varepsilon_{rst}\, \lambda^s\, \nu^t$$

$$=\; \lambda_p\, (\, \delta^{pr}_{lm}\, \mu^l\, \lambda^m\,)\, \varepsilon_{rst}\, \lambda^s\, \nu^t \;-\; (\, \lambda_p\, \lambda^p\,)\, \varepsilon_{rst}\, \mu^r\, \lambda^s\, \nu^t$$

$$=\; (\lambda_p\, \mu^p\,)\, \varepsilon_{rst}\, \lambda^r\, \lambda^s\, \nu^t \;-\; (\lambda_p\, \lambda^p\,)\, \varepsilon_{rst}\, \lambda^r\, \mu^s\, \nu^t$$

$$=\; -\; \varepsilon_{rst}\, \mu^r\, \lambda^s\, \nu^t \;=\; \varepsilon_{rst}\, \lambda^r\, \mu^s\, \nu^t\, .$$

Therefore

$$\sin b \sin c \sin \mathrm{A} \;=\; \varepsilon_{rst}\, \lambda^r\, \mu^s\, \nu^t\, .$$

By permuting λ, μ and ν on the right hand side, we get expressions which are unchanged except for the order of the terms. On the other hand, two new expressions are generated on the left. Hence

$$\sin b \sin c \sin \mathrm{A} \;=\; \sin c \sin a \sin \mathrm{B} \;=\; \sin a \sin b \sin \mathrm{C}\, .$$

In the form

$$\frac{\sin a}{\sin \mathrm{A}} \;=\; \frac{\sin b}{\sin \mathrm{B}} \;=\; \frac{\sin c}{\sin \mathrm{C}}\, ,$$

this is the **law of sines of spherical trigonometry**.

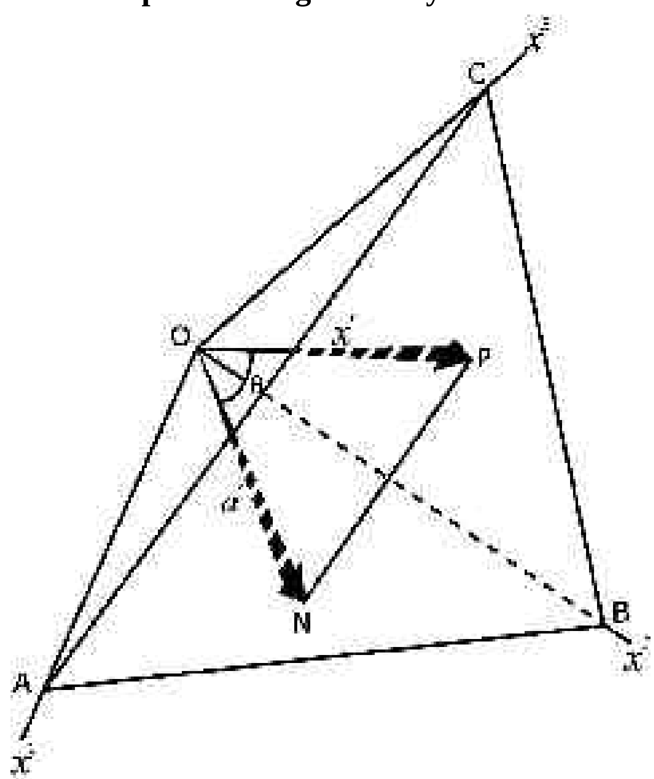


Figure 64

3. Points, Lines, and Planes

Consider a rectilinear coordinate system as in Fig. 64, with origin at O. Let A, B and C be the intercepts of a plane ABC on the respective x^1-, x^2- and x^3-axes. Let a^r be the vector from O to the plane ABC and normal to it. Then the vector x^r to any point P in the plane must satisfy the equation

$$a_r x^r = |a_r| \cdot |x^r| \cdot \cos\theta = a\, d \cos\theta ,$$

where $d = |x^r|$. But from Fig. 64 it is clear that $d \cos\theta = a$, whence

$$a_r x^r = (a)^2 , \quad \left(\frac{a_r}{a} \right) x^r = a$$

or, putting $a_r / a = n_r$, a unit normal,

$$n_r x^r = a . \tag{3.1}$$

This equation of the plane in three dimensions is formally similar to the equation of the straight line in two dimensions. For a fixed vector a_r, the vectors x^r generate the plane whose unit normal is n_r and whose distance from the origin at its nearest point is a. On the other hand, if x^r is held fixed, the set of all a_r which satisfy equation (3.1) will define all planes which intersect in the point whose coordinates are x^r. We thus speak of a_r as the coordinates of the plane ABC just as the x^r are the coordinates of the point P. We may generate all planes parallel to a given plane by varying a from $-\infty$ to $+\infty$.

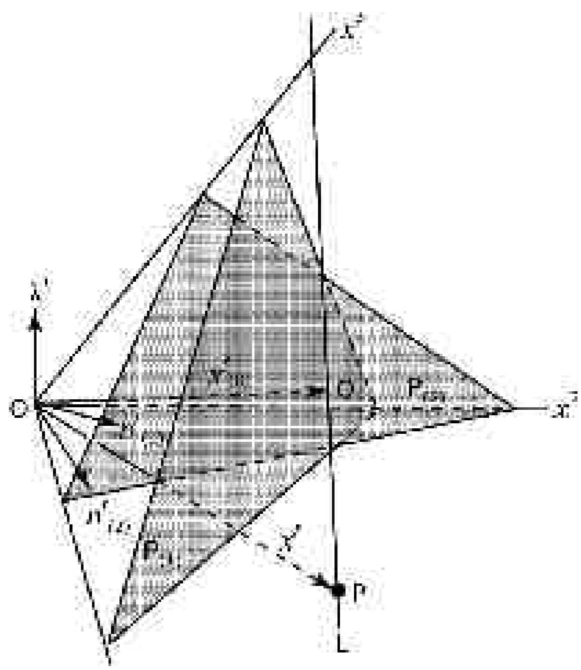


Figure 65

If we consider two planes $\mathrm{P}_{(1)}$ and $\mathrm{P}_{(2)}$ (see Fig. 65) whose normals are neither parallel nor antiparallel, they must intersect in a line. For any point on this line it is necessarily true that

(3.2) $$n_r^{(1)} x^r = a^{(1)}, \quad n_r^{(2)} x^r = a^{(2)}.$$

These are two linear equations in three unknowns. They therefore imply that in general a one-parameter linear relation must exist among common coordinates (components) x^r. Without loss of generality, we may suppose this relation to be of the parametric form

(3.3) $$x^r = x_{(0)}^r + l\lambda^r,$$

where λ^r is a unit vector and l is the parameter. Now the vector $x^r = x_{(0)}^r$, corresponding to the value $l = 0$ of the parameter, must satisfy the equations

(3.2). Hence

(3.4) $$n_r^{(1)} x_{(0)}^r = a^{(1)} \quad \text{and} \quad n_r^{(2)} x_{(0)}^r = a^{(2)}.$$

Therefore, multiplying equation (3.3) by $n_r^{(1)}$ and using equations (3.2) and (3.4),

$$n_r^{(1)} x^r = a^{(1)} = n_r^{(1)} x_{(0)}^r + l(n_r^{(1)} \lambda^r) = a^{(1)} + l(n_r^{(1)} \lambda^r),$$

we see that $n_r^{(1)} \lambda^r = 0$, or $n_r^{(1)}$ and λ^r are orthogonal. In a similar way, we can show that $n_r^{(2)}$ and λ^r are orthogonal. Therefore

(3.5) $$\lambda^r = \frac{\varepsilon^{rst} n_s^{(1)} n_t^{(2)}}{|\varepsilon^{ijk} n_j^{(1)} n_k^{(2)}|}.$$

Let us now multiply both sides of equation (3.3) by λ_r. This gives

$$\lambda_r x^r = \lambda_r x_{(0)}^r + l(\lambda_r \lambda^r) = \lambda_r x_{(0)}^r + l.$$

If we choose $x_{(0)}^r$ to be the point on the line nearest the origin, then

(3.6) $$\lambda_r x_{(0)}^r = 0$$

and $l = \lambda_r x^r$ is simply the distance from $x_{(0)}$ along the line. We see also that λ^r is orthogonal to $x_{(0)}^r$.

Ex. (3.1) (a) Identify the plane whose equation is

$$2x^1 + x^2 - 3x^3 = 6$$

in a Cartesian coordinate system. (b) Do the same for the plane whose equation is

$$-3x^1 + 4x^2 + x^3 = 12.$$

(c) find the equation of the line in which they intersect. (Hint: $x_{(0)}^r$ is a solution of equations (3.2) and (3.6).)

Ans. (a) The vector $\mathrm{N}_r^{(1)} = (2, 1, -3)$ *has magnitude* $\mathrm{N}^{(1)} = \sqrt{14}$. *Hence the equation of the plane in canonical form is*

$$\sqrt{14}\left(\frac{x^1}{7}+\frac{x^2}{14}-\frac{3x^3}{14}\right)=\frac{3\sqrt{14}}{7}.$$

Therefore the unit normal to the plane is

$$n_r^{(1)}=\mathrm{N}_r^{(1)}/\mathrm{N}^{(1)}=\sqrt{14}\left(\frac{1}{7},\frac{1}{14},\frac{-3}{14}\right).$$

The distance from the origin to the nearest point of the plane is $a^{(1)}=3\sqrt{14}/7$ *units.*

(b) The vector $\mathrm{N}_r^{(2)}=(-3,4,1)$ *has a magnitude* $\mathrm{N}^{(2)}=\sqrt{26}$. *Hence the unit normal to the plane is*

$$n_r^{(2)}=\frac{\sqrt{26}}{26}(-3,4,1).$$

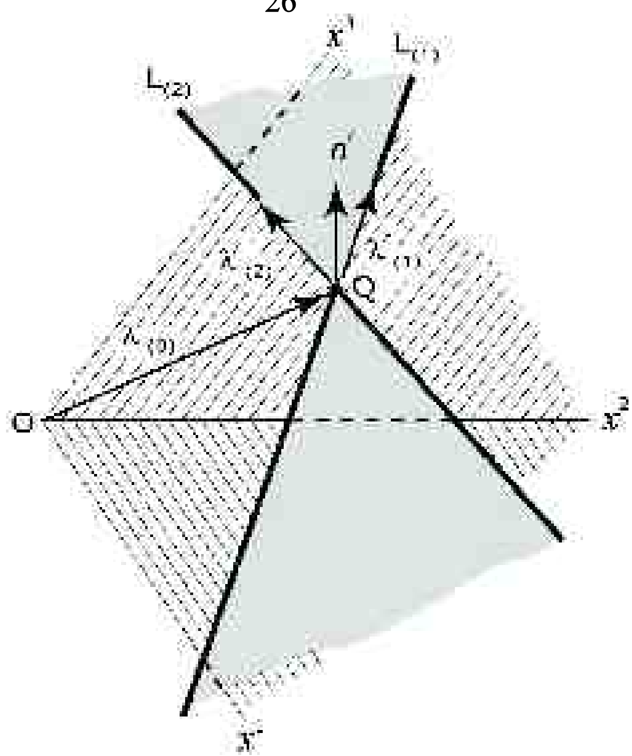


Figure 66

The distance from the origin to the nearest point of the plane is evidently

$$a^{(2)}=\frac{6\sqrt{26}}{13}.$$

(c) The equation of the line of intersection is the equation (3.3) with

$$\lambda^r=\frac{\sqrt{339}}{339}(13,7,11),\quad x_{(0)}^r=\frac{1}{113}(-54,336,-150).$$

We have seen how to determine the equation of the line which is formed by the intersection of two planes. Let us now consider the converse problem: What is the plane containing two given intersecting straight lines? We suppose the lines to have the equations

$$\text{(3.7)} \qquad L_{(1)} : x^r = x^r_{(1)} + l\,\lambda^r_{(1)} \quad \text{and} \quad L_{(2)} : x^r = x^r_{(2)} + m\,\lambda^r_{(2)},$$

where the x^r are the components of the position vector to any point on the line, l and m are the respective parameters of the lines, $x^r_{(1)}$ and $x^r_{(2)}$ are the position vectors of a particular point on the line (conveniently the point nearest the origin), and $\lambda^r_{(1)}$ and $\lambda^r_{(2)}$ the unit vectors in the directions of the lines. To be certain that the two lines are not identical or parallel, we require that

$$|\,\varepsilon_{ijk}\,\lambda^j_{(1)}\,\lambda^k_{(2)}\,| \neq 0 .$$

Suppose that $x^r_{(0)}$ is the position vector to the point of intersection of the two lines. Then clearly $x^r - x^r_{(0)}$, if x^r is in the plane containing the two lines, must be some linear combination of $\lambda^r_{(1)}$ and $\lambda^r_{(2)}$. Hence

$$\text{(3.8)} \qquad x^r = x^r_{(0)} + \alpha\,\lambda^r_{(1)} + \beta\,\lambda^r_{(2)}$$

is the equation of this plane. The two arbitrary parameters α and β are the parameters of the plane.

To put this equation for the plane into the form of equation (3.1), we first form the unit normal

$$\text{(3.9)} \qquad n_r = \frac{\varepsilon_{rst}\,\lambda^s_{(1)}\,\lambda^t_{(2)}}{|\,\varepsilon_{ijk}\,\lambda^j_{(1)}\,\lambda^k_{(2)}\,|} .$$

Multiplying equation (3.8) through by n_r, we see that

$$n_r x^r = a = n_r x^r_{(0)} .$$

This is the desired result.

Ex. (3.2) Show that the plane defined by two lines whose equations are given in equation (3.7) could equally well have equations of the form

$$x^r = x^r_{(1)} + \alpha\,\lambda^r_{(1)} + \beta\,\lambda^r_{(2)} \quad \text{or} \quad x^r = x^r_{(2)} + \alpha\,\lambda^r_{(1)} + \beta\,\lambda^r_{(2)} .$$

Thus, show that the condition for the intersection of the two lines is that

$$n_r x^r_{(1)} = a = n_r x^r_{(2)} .$$

(Hint: Both $x^r_{(1)}$ and $x^r_{(2)}$ must lie in the plane and $\lambda^r_{(1)}$ and $\lambda^r_{(2)}$ will in any case span the plane.)

4. Cones and Quadrics

We have seen that in general, an equation of the form

$$a_i x^i = b$$

is the equation of a straight line in rectilinear coordinates in two dimensions; in three dimensions it is more generally the equation of a plane. We have also seen that in two dimensions, an equation of the form

$$t_{ij} x^i x^j = b$$

is in general the equation of an ellipse or hyperbola $(b \neq 0)$ or a pair of straight lines $(b = 0)$. We now inquire what such an equation represents in three dimensions.

First, we recognize that there is no loss of generality in supposing that t_{ij} is symmetric, for if it is not, it may be resolved as

$$t_{ij} = \frac{1}{2}\left(t_{ij} + t_{ji}\right) + \left(t_{ij} - t_{ji}\right) = s_{ij} + a_{ij}$$

into a symmetric part s_{ij} and an antisymmetric part a_{ij}. Then

$$t_{ij} x^i x^j = s_{ij} x^i x^j + a_{ij} x^i x^j .$$

But
$$a_{ij} x^i x^j = a_{ji} x^j x^i = -a_{ij} x^j x^i = -a_{ij} x^i x^j ,$$

hence it is zero.

We therefore proceed by asking the question: Is there some vector or vectors v^i such that, given a symmetric tensor s_{ij}, there is a constant λ such that $s_j{}^i \mathrm{v}^j = \lambda \mathrm{v}^i = \lambda \delta^i_j \mathrm{v}^j$?

If so, we must have

$$(s_j{}^i - \lambda \delta^i_j) \mathrm{v}^i = 0 ,$$

a set of equations whose only solution is $\mathrm{v}^i = 0$ unless

$$3!\,|s_j{}^i - \lambda \delta^i_j| = \delta^{ijk}_{lmn} (s_i{}^l - \lambda \delta^l_i)(s_j{}^m - \lambda \delta^m_j)(s_k{}^n - \lambda \delta^n_k) = 0 . \tag{4.1}$$

Expanded, this equation gives a cubic in λ, namely

$$\delta^{ijk}_{lmn} [s_i{}^l s_j{}^m s_k{}^n - \lambda (s_i{}^l s_j{}^m \delta^n_k + s_i{}^l \delta^m_j s_k{}^n + \delta^l_i s_j{}^m s_k{}^n)$$

$$+ \lambda^2 (s_i{}^l \delta^m_j \delta^n_k + \delta^l_i s_j{}^m \delta^n_k + \delta^l_i \delta^m_j s_k{}^n) - \lambda^3 \delta^l_i \delta^m_j \delta^n_k]$$

$$= \delta^{ijk}_{lmn} s_i{}^l s_j{}^m s_k{}^n - \lambda (\delta^{ij}_{lm} s_i{}^l s_j{}^m + \delta^{ik}_{ln} s_i{}^l s_k{}^n + \delta^{jk}_{mn} s_j{}^m s_k{}^n) .\beta$$

$$+ 2\lambda^2 (\delta^i_l s_i{}^l + \delta^j_m s_j{}^m + \delta^k_n s_k{}^n) - \lambda^3 \delta^{ijk}_{ijk}$$

$$= \delta^{ijk}_{lmn} s_i{}^l s_j{}^m s_k{}^n - 3\lambda (\delta^{ij}_{lm} s_l{}^i s_j{}^m) + 6\lambda^2 s_i{}^i - 6\lambda^3 = 0$$

Since the coefficients of the powers of λ are all invariants, the roots of this equation must be also. We know that at least one root must be real inasmuch as every cubic polynomial with real coefficients must have at least one real root.

Let us write equation (4.1) in the form

$$|s_{ij} - \lambda g_{ij}| = 0 .$$

We can now show that not only are the roots λ of the **determinantal equation** invariants, but ***all*** are real if s_{ij} is symmetric. To do so, suppose that one of the roots $\lambda = a + ib$ is complex. Then it would be true that

$$[s_{mn} - (a + ib) g_{mn}] \mathrm{v}^n = 0$$

for some necessarily complex eigenvector $\mathrm{v}^{\mathrm{n}} = \lambda^n + i \mu^n$. That is,

$$\begin{aligned} &[s_{mn} - (a + ib) g_{mn}] (\lambda^n + i \mu^n) \\ &= (s_{mn} \lambda^n - a g_{mn} \lambda^n + b g_{mn} \mu^n) \\ &\quad + i(- b g_{mn} \lambda^n + s_{mn} \mu^n + a g_{mn} \mu^n - a g_{mn} \mu^n) = 0 . \end{aligned}$$

Therefore

$$(4.2) \qquad \begin{aligned} (s_{mn} - a g_{mn}) \lambda^n + b g_{mn} \mu^n &= 0 , \quad \text{and} \\ b g_{mn} \lambda^n - (s_{mn} - a g_{mn}) \mu^n &= 0 . \end{aligned}$$

Multiplying the former by μ^m, the latter by λ^m and subtracting, we get

$$b (g_{mn} \mu^m \mu^n + g_{mn} \lambda^m \lambda^n) = b [|\mu^m|^2 + |\lambda^m|^2] = 0 .$$

Since not both $|\mu^m|^2$ and $|\lambda^m|^2$ can be zero, it follows that $b = 0$, whence $\lambda = a$ is real. Note that this proof is independent of the fact that the number of dimensions is three.

We may note also, for the sake of future applications (as to moment of inertia tensors), that if $s_{mn} \mathrm{v}^m \mathrm{v}^n > 0$ for all non-zero v^m, then from equation (4.2), multiplying the first by λ^m and the second by μ^m and subtracting, we get

$$s_{mn} \lambda^m \lambda^n + s_{mn} \mu^m \mu^n = a (g_{mn} \lambda^m \lambda^n + g_{mn} \mu^m \mu^n) .$$

Since both the left hand side and the parenthesis on the right hand side are positive, so also is a. In this case, the roots are not only real invariants, but positive as well. When s_{mn} satisfies the condition $s_{mn} \mathrm{v}^m \mathrm{v}^n > 0$ for all non-zero v^m, it is said to be **positive definite**.

We have therefore shown that the equation

$$(4.3) \qquad s_{ij} \mathrm{v}^j = \lambda \mathrm{v}_i$$

has a solution $v^i_{(k)}$ for three values of $\lambda_{(k)}$, the **eigenvalues**. These solutions $v^i_{(k)}$, $(k = 1, 2, 3)$ are the **eigenvectors**. As before, we can show that these eigenvectors are mutually orthogonal if the eigenvalues $\lambda_{(k)}$ are distinct. Thus

$$s_{ij} v^j_{(k)} = \lambda_{(k)} v^{(k)}_i, \quad s_{ij} v^j_{(l)} = \lambda_{(l)} v^{(l)}_i,$$

$$s_{ij} v^j_{(k)} v^i_{(l)} = \lambda_{(k)} v^{(k)}_i v^i_{(l)}, \quad s_{ij} v^j_{(l)} v^i_{(k)} = \lambda_{(l)} v^{(l)}_i v^i_{(k)}.$$

But because $s_{ij} = s_{ji}$,

$$\lambda_{(k)} v^{(k)}_i v^i_{(l)} = s_{ij} v^j_{(k)} v^i_{(l)} = s_{ji} v^i_{(k)} v^j_{(l)} = s_{ij} v^i_{(k)} v^j_{(l)} = \lambda_{(l)} v^{(l)}_i v^i_{(k)},$$

$$(\lambda_{(k)} - \lambda_{(l)}) v^{(k)}_i v^i_{(l)} = 0.$$

Since we postulated that $\lambda_{(k)} - \lambda_{(l)} \neq 0$, it follows that $v^{(k)}_i v^i_{(l)} = 0$. The eigenvectors are therefore orthogonal.

Ex. (4.1) Find the eigenvalues and the eigenvectors of the symmetric tensor

$$s_{ij} = \frac{1}{4225} \left\| \begin{matrix} 9121 & -2040 & 1872 \\ -2040 & 5075 & -780 \\ 1872 & -780 & 11{,}154 \end{matrix} \right\|,$$

where the fraction $1/4225$ *multiplies* ***each*** *component within the matrix. Take the coordinate system to be Cartesian.*

Ans. The determinantal equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

The roots of this equation are $\lambda = 1, 2, 3$. *When* $\lambda = 1$, *the solution of the equation is*

$$v^i_{(1)} = (5\alpha, 12\alpha, 0),$$

α *arbitrary. The other eigenvectors are*

$$v_{(2)} = (-48\beta, 20\beta, 39\beta), \quad (\lambda = 2)$$

$$v^i_{(3)} = (36\gamma, -15\gamma, 52\gamma), \quad (\lambda = 3).$$

Ex. (4.2) Show that the eigenvectors of Ex. (4.1) are mutually orthogonal.

This being the case, we may adopt the directions of the eigenvectors, the invariant directions, as the directions of a set of orthogonal axes. In this reference system we have

$$v^i_{(1)} = (v_{(1)}, 0, 0), \quad v^i_{(2)} = (0, v_{(2)}, 0), \quad \text{and} \quad v^i_{(3)} = (0, 0, v_{(3)}).$$

By substitution of these vectors into equation (4.3), we can show at once that

$$s_{ij} = \lambda_{(i)} g_{ij},$$

where $g_{ij} = 0$ if $i \neq j$ because the coordinate axes are orthogonal. In other words, the symmetric tensor s_{ij} becomes a diagonal tensor in the reference frame of the invariant directions.

It is now very easy to see what the equation

$$t_{ij} x^i x^j = s_{ij} x^i x^j = b \tag{4.4}$$

represents. In the frame of reference of the invariant directions, it is simply

$$\frac{(x^1)^2}{a_{(1)}^2} + \frac{(x^2)^2}{a_{(2)}^2} + \frac{(x^3)^2}{a_{(3)}^2} = 1,$$

where $a_{(i)}^2 = b / \lambda_{(i)}$. The position vectors $p^i = (x^1, x^2, x^3)$ which satisfy this equation evidently define a quadric provided not all $a_{(i)}^2$ are negative. The semi-axes of the quadric are equal to $|a_{(i)}|$.

If $b = 0$ but the $\lambda_{(i)}$ are not all of the same sign, then clearly equation (4.4) is that of a cone. If the $\lambda_{(i)}$ are all of the same sign, the equation (4.4) can be satisfied only by $p^i = 0$.

Ex. (4.3) Interpret the equation

$$s_{ij} x^i x^j = 1,$$

where s_{ij} has the Cartesian components given in Ex. (4.1).

Ans. This equation represents an ellipsoid with axes $a = 1$, $b = \sqrt{2}/2$, and $c = \sqrt{3}/3$ along the directions of the eigenvectors.

5. The Kinematics of Rigid Bodies

A principal application of vectors and tensors in rectilinear coordinates is to the motion of rigid bodies. We define a rigid body as a set of particles whose mutual distances are fixed. Where such a conception need be made more refined, summation over particles may be replaced by integration over mass elements. With no essential loss of generality, however, we may treat the dynamics of rigid bodies as that of a set of rigidly connected particles.

Let x^s be a set of rectilinear coordinates in an inertial frame of reference. Let the particle of mass $m_{(i)} > 0$ have coordinates $x_{(i)}^s$; that is, let the particle $m_{(i)}$ have a position vector whose components are $x_{(i)}^s$. We now define the center of gravity of the set of particles (see Fig. 67) to be the point C whose position vector ξ^r is given by

$$\sum_i m_{(i)} x_{(i)}^r = \xi_r \sum_i m_{(i)} = m \xi^r. \tag{5.1}$$

With respect to the center of gravity, the ith particle has a position vector

$$y_{(i)}^r = x_{(i)}^r - \xi^r.$$

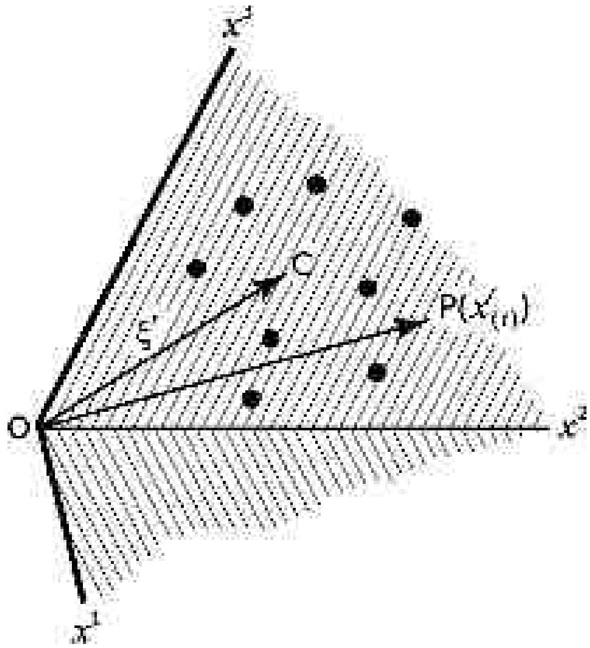


Figure 67

From equation (5.1) it follows that

$$\sum_i m_{(i)} y^r_{(i)} = 0 . \tag{5.2}$$

Ex. (5.1) Prove equation (5.2).

Ex. (5.2) Given three particles with masses $m_{(1)} = 2$, $m_{(2)} = 7$, $m_{(3)} = 3$. *If they are at points whose Cartesian position vectors are* $x^i_{(1)} = (2, 0, 3)$, $x^i_{(2)} = (-4, 1, 6)$, *and* $x^i_{(3)} = (3, -5, 2)$, *find the position vector of the center of gravity and the respective position vectors relative to the center of gravity. Show that the weighted mean of the latter is zero.*

Ans. $\xi^i = \left(-\dfrac{5}{4}, -\dfrac{2}{3}, \dfrac{9}{2} \right)$.

Hence

$$y_{(1)} = \left(\frac{13}{4}, \frac{2}{3}, -\frac{3}{2} \right),\ y^i_{(2)} = \left(-\frac{11}{4}, \frac{5}{3}, \frac{3}{2} \right),\ y_{(3)} = \left(\frac{17}{4}, -\frac{13}{3}, -\frac{5}{2} \right).$$

The rigid body may be in some kind of motion. Then each particle will have a velocity

$$\mathrm{v}^r_{(i)} = \frac{dx^r_{(i)}}{dt} = \dot{x}^r_{(i)} .$$

The velocity of the center of gravity will be $\dot{\xi}^r$. With respect to the center of gravity, the velocity of each particle will be

$$\frac{dy^r_{(i)}}{dt} = \dot{y}^r_{(i)} - \dot{\xi}^r = w^r_{(i)}.$$

From equation (5.2) it is clear that

$$\sum_i m_{(i)} w^r_{(i)} = 0. \tag{5.3}$$

Between any two particles i and j there is a fixed distance

$$d_{(ij)} = \left[g_{rs} (x^r_{(i)} - x^r_{(j)}) (x^s_{(i)} - x^s_{(j)}) \right]^{1/2}.$$

Since this is a constant distance within a rigid body, its time derivative is zero, whence

$$d_{(ij)} \frac{d}{dt} d_{(ij)} = g_{rs} (x^r_{(i)} - x^r_{(j)}) (\mathrm{v}^s_{(i)} - \mathrm{v}^s_{(j)}) = 0. \tag{5.4}$$

But clearly

$$x^s_{(i)} - x^s_{(j)} = y^s_{(i)} - y^s_{(j)},$$

which implies that

$$\mathrm{v}^s_{(i)} - \mathrm{v}_{(j)} = w^s_{(i)} - w^s_{(j)}.$$

Therefore equation (5.4) becomes

$$g_{rs} (y^r_{(i)} - y^r_{(j)}) (w^s_{(i)} - w^s_{(j)}) = 0 \text{ or}$$

$$g_{rs} [y^r_{(i)} w^s_{(i)} + y^r_{(j)} w^s_{(j)}] = g_{rs} [y^r_{(i)} w^s_{(j)} + y^s_{(j)} w^s_{(i)}] \tag{5.5}$$

for all i and j. Now for fixed j let this equation be multiplied by $m_{(i)}$ and summed over i. The result will be

$$g_{rs} \left[\sum_i m_{(i)} y^r_{(i)} w^r_{(i)} + y^r_{(j)} w^s_{(j)} \sum_i m_{(i)} \right]$$
$$= g_{rs} \left[w^s_{(j)} \sum_i m_{(i)} y^r_{(i)} + y^r_{(j)} \sum_i m_{(i)} w^s_{(i)} \right].$$

By equations (5.2) and (5.3), the sums on the right hand side are zero. Hence

$$g_{rs} \left[m\, y^r_{(j)} w^s_{(j)} + \sum_i m_{(i)} y^r_{(i)} w^s_{(i)} \right] = 0. \tag{5.6}$$

Multiply this equation by $m_{(j)}$ and sum over j. We then have

$$g_{rs}\left[m\sum_j m_{(j)}y^r_{(j)}w^s_{(j)} + \left(\sum_i m_{(i)}y^r_{(i)}w^s_{(i)}\right)\sum_j m_{(j)}\right]$$
$$= m\,g_{rs}\left[\sum_j m_{(j)}y^r_{(j)}w^s_{(j)} + \sum_i m_{(i)}y^r_{(i)}w^s_{(i)}\right] = 2\,m\,g_{rs}\sum_i y^r_{(i)}w^s_{(j)} = 0\,.$$

Since this is to be true whatever values may be assigned the $m_{(i)}$, it follows that

$$g_{rs}y^r_{(i)}w^2_{(i)} = 0 \tag{5.7}$$

for each i. This implies that the left hand side of equation (5.5) vanishes, whence from the right hand side

$$g_{rs}\left[y^r_{(j)}w^s_{(i)} + y^r_{(i)}w^s_{(j)}\right] = 0 \tag{5.8}$$

for all i and j.

Let us note in passing that from equation (5.7) we can infer also that

$$\frac{d}{dt}\left(d^2_{(j)}\right) = \frac{d}{dt}\left[g_{rs}y^r_{(j)}y^s_{(j)}\right] = 2\,g_{rs}y^r_{(j)}w^s_{(j)} = 0\,.$$

This implies that the distance $d_{(j)}$ of particle j from the center of gravity does not change; i.e., the center of gravity is at a fixed point within the rigid body.

Let us now examine equation (5.8), which we write as

$$y^r_{(j)}w^{(i)}_r = -\,y^r_{(i)}w^{(j)}_r\,.$$

Now for any i this must be true for every j, and vice versa. Hence $w^s_{(i)}$ must be a linear homogeneous function of $y^s_{(i)}$, the *same* linear function that $w^s_{(j)}$ is of $y^s_{(j)}$. That is, we must have

$$w^{(i)}_r = \boldsymbol{\omega}_{rs}y^s_{(i)}\,,\quad w^{(j)}_r = \boldsymbol{\omega}_{rs}y^s_{(j)}\,, \tag{5.9}$$

where $\boldsymbol{\omega}_{rs}$ is a covariant second rank tensor. Substituting equation (5.9) into equation (5.8), it becomes

$$\begin{aligned}&\boldsymbol{\omega}_{rs}y^r_{(j)}y^s_{(i)} + \boldsymbol{\omega}_{rs}y^r_{(i)}y^s_{(j)}\\ &= \boldsymbol{\omega}_{rs}y^r_{(j)}y^s_{(i)} + \boldsymbol{\omega}_{sr}y^s_{(i)}y^r_{(j)} = (\boldsymbol{\omega}_{rs} + \boldsymbol{\omega}_{sr})\,y^r_{(j)}y^s_{(i)} = 0\,.\end{aligned} \tag{5.10}$$

In general, this requires that

$$\boldsymbol{\omega}_{rs} = -\,\boldsymbol{\omega}_{sr}\,. \tag{5.11}$$

Hence $\boldsymbol{\omega}_{rs}$ is an *antisymmetric* covariant tensor of the second rank. It is called the **angular velocity tensor**.

The dual of the angular velocity tensor is the **angular velocity vector**

$$\omega^r = \frac{1}{2}\varepsilon^{rmn}\omega_{mn}, \text{ whence } \omega_{mn} = \varepsilon_{mnr}\omega^r. \tag{5.12}$$

To give ω^r a geometrical interpretation, we consider a point within the rigid body whose position vector has components $y^s_{(k)} = \alpha\,\omega^s$, where α is some convenient invariant parameter. Then by equation (5.9), the velocity of the particle k at point $y^s_{(k)}$ is

$$\omega^{(k)}_r = \omega_{rs}y^s_{(k)} = \alpha\,\omega_{rs}\omega^s = \varepsilon_{rst}\omega^s\omega^t = 0.$$

In other words, the line defined by the equation $y^s_{(k)} = \alpha\,\omega^s$, lying along ω^s, is at rest relative to the center of gravity. It is called the **instantaneous axis of rotation**. The magnitude of ω^s is called the **angular velocity**.

We note finally that when $\omega^s = 0$, we must also have $\omega_{mn} = 0$. Then

$$\omega^{(k)}_s = \mathrm{v}^{(k)}_s - \dot{\xi}_s = \omega_{sr}y^r_{(k)} = 0, \quad \mathrm{v}^{(k)}_s = \dot{\xi}_s, \quad \mathrm{v}^s_{(k)} = \dot{\xi}^s,$$

for all k. Such a motion is called a **pure translation**.

*Ex. (5.3) Given an angular velocity vector whose Cartesian components are $\omega^r = (3, 1, \sqrt{6})$. (a) What is the magnitude of the angular velocity vector? (This is usually termed simply the **angular velocity**.) (b) What angles does it make with the coordinate axes? (c) What are the components of the angular velocity tensor?*

Ans. (a) $|\omega^r| = 4$. The units are angle per unit time. Most commonly, this is radians/second, or some similar unit. (b) Since

$$\omega^r = 4\left(\frac{3}{4}, \frac{1}{4}, \frac{\sqrt{6}}{4}\right),$$

where the quantities in parentheses are the components of a unit vector, we have

$$\theta_1 = \cos^{-1}\frac{3}{4} = 41°\,25', \ \theta_2 = \cos^{-1}\frac{1}{4} = 75°\,31', \ \theta_3 = \cos^{-1}\frac{\sqrt{6}}{4} = 52°\,14'.$$

(*c*)
$$\omega_{mn} = \left\Vert \begin{matrix} 0 & \sqrt{6} & -1 \\ -\sqrt{6} & 0 & 3 \\ 1 & -3 & 0 \end{matrix} \right\Vert .$$

Ex. (5.4) If ω_r is the angular velocity vector, then all vectors y^r which satisfy the condition

$$\omega_r y^r = 0 \tag{5.13}$$

*define the plane of the **equator** of the rigid body. Since* $y^r = 0$ *satisfies this equation, it is clear that the center of gravity lies in the plane of the equator. The equation also indicates that the equator and the axis of rotation are mutually orthogonal.*

Show that for any point y^r *in the plane of the equator the angular velocity tensor is given by the expression*

$$\omega_{pq} = \frac{\delta^{st}_{pq} w_s y_t}{|y^k|^2} . \tag{5.14}$$

Ans. Since $w_p = \omega_{pq} y^q = \varepsilon_{pqr} y^q \omega^r$, *it is clear that* w^p *and* ω^p *are orthogonal. By equation (5.13),* y^p *and* w^p *are likewise. Hence* y^p, w^p *and* ω^p *form an orthogonal triad, so that*

$$\omega^r = k \varepsilon^{rst} w_s y_t , \tag{5.15}$$

where k *is an invariant factor to be determined. The value of* k *may be fixed by noting that*

$$\begin{aligned} w_p &= \omega_{pq} y^q = \varepsilon_{pqr} y^q \omega^r = k \varepsilon_{pqr} \varepsilon^{rst} w_s y_t y^q \\ &= k (\delta^{st}_{pq} w_s y_t) y^q = k [w_p y_q - w_q y_p] y^q \\ &= k [(y_q y^q) w_p - (w_q y^q) y_p] . \end{aligned}$$

Since $w_q y^q = 0$, *we may take this as a true equation by choosing* $k (y_q y^q) = 1$, *whence equation (5.14) follows from equations (5.12) and (5.15).*

Ex. (5.5) Show that the axis of rotation is defined by the location of any two points on the equator which are not collinear with the center of gravity.

Ans. If the two points are $y^r_{(1)}$ *and* $y^r_{(2)}$, *then*

$$\omega_r y^r_{(1)} \text{ and } \omega_r y^r_{(2)} = 0$$

provide two independent equations from which the ratios $\omega_1 : \omega_2 : \omega_3$ *may be found provided* $y^r_{(1)} \neq k y^r_{(2)}$, k *constant.*

Ex. (5.6) Given that in a Cartesian coordinate system at time $t = 0$: *(a)* $m_{(1)} = 3$, $x^i_{(1)} = (1 , -21 , 17)$, $m_{(2)} = 5$, $x^i_{(2)} = (-7 , 3 , 1)$. *Find the position vector of the center of gravity. (b) If the particles have respective velocities* $\dot{x}^i_{(1)} = \mathrm{v}^i_{(1)} = (-45 , 17 , 26)$ *and* $\dot{x}^i_{(2)} = \mathrm{v}^i_{(2)} = (43 , 9 , -30)$, *find the velocity of the center of gravity. (c) What are the particles' position vectors relative to the center of gravity? (d) What are the particles' velocities relative to the center of gravity? (e) What is the angular velocity tensor of the system? (Hint: use equation (5.14).) (f) What is the angular velocity vector for the system? (Use equation (5.12).) (g) Check your answer to (f) by using equation (5.15).*

Ans. $(a)\ \xi^i = (-4, -6, 7)$; $(b)\ \dot{\xi}^i = (10, 12, -9)$;

$(c)\ y^i_{(1)} = (5, -15, 10),\ y^i_{(2)} = (-3, 9, -6)$;

$(d)\ w^i_{(1)} = (-55, 5, 35),\ w^i_{(2)} = (33, -3, -21)$;

$$(e)\ \omega_{pq} = \left\| \begin{matrix} 0 & \frac{16}{7} & -\frac{29}{14} \\ -\frac{16}{7} & 0 & \frac{23}{14} \\ \frac{29}{14} & -\frac{23}{14} & 0 \end{matrix} \right\| ; \quad (f)\ \omega^r = \left(\frac{23}{14}, \frac{29}{14}, \frac{16}{7} \right).$$

6. The Dynamics of Rigid Bodies

We have already seen that the velocity of every constituent particle i of a rigid body is expressible as

$$\dot{x}_r^{(i)} = \mathrm{v}_r^{(i)} = \dot{\xi}_r + w_r^{(i)} = \dot{\xi}_r + \varepsilon_{rst}\,\omega^s y^t_{(i)}, \tag{6.1}$$

where $\dot{\xi}_r$ and ω_{rs} are independent of i and therefore of $y^r_{(i)}$. Here $\dot{\xi}_r$ is the velocity of translation and ω^s is the angular velocity vector.

Let us now write the equations of motion for ***any*** particle, whether or not it is an element of a rigid body; indeed, it could as well be a free particle. Then according to Newton's second law,

$$m_{(i)} \frac{d\mathrm{v}^r_{(i)}}{dt} = \mathrm{F}^r_{(i)}, \tag{6.2}$$

where $\mathrm{F}^r_{(i)}$ is the force acting upon particle i. Writing a similar equation for each particle and summing both sides over all particles, one obtains

$$\sum_i m_{(i)} \frac{d\mathrm{v}^r_{(i)}}{dt} = \sum_i \mathrm{F}^r_{(i)} = \mathrm{F}^r.$$

(Note that $\mathrm{F}^r_{(i)}$ is applied to the ith particle and that the sum is therefore over vectors at different points. This is legitimate only in a rectilinear coordinate system, where a vector may be transported parallel to itself to any new position without any change in the values of the components.)

From equation (5.1) it therefore follows that

$$m \frac{d\dot{\xi}^r}{dt} = \mathrm{F}^r. \tag{6.3}$$

In other words, the center of gravity of a system of particles — *any* system — moves as though it were a particle of the same total mass as the system, acted upon by a net force equal to the total sum of all the forces, applied at the center of gravity. Since rigid bodies have not been excluded, this is true of rigid bodies as well as any other system of particles.

Let us now multiply both sides of equation (6.2) by $\varepsilon_{str}\, x^t_{(i)}$ and sum over i. *Then, again for any system of particles,*

$$\sum_i m_{(i)}\, \varepsilon_{str}\, x^t_{(i)} \frac{d\mathrm{v}^r_{(i)}}{dt} = \sum_i \varepsilon_{str}\, x^t_{(i)}\, \mathrm{F}^r_{(i)} .$$

Now since $$\varepsilon_{str}\, \mathrm{v}^t_{(i)}\, \mathrm{v}^r_{(i)} = \varepsilon_{str}\, \dot{x}^t_{(i)}\, \dot{x}^r_{(i)} = 0 ,$$

the preceding equation may be written in the form

$$\frac{d\mathrm{A}_s}{dt} = \frac{d}{dt}\left[\sum_i m_{(i)}\, \varepsilon_{str}\, x^t_{(i)}\, \dot{x}^r_{(i)}\right] = \sum_i \varepsilon_{str}\, x^t_{(i)}\, \mathrm{F}^r_{(i)} = \mathrm{L}_s . \tag{6.4}$$

The quantity

$$\mathrm{A}_s = \sum_i m_{(i)}\, \varepsilon_{str}\, x^t_{(i)}\, \dot{x}^r_{(i)} \tag{6.5}$$

is called the **angular momentum** of the system of particles. The quantity

$$\mathrm{L}_s = \sum_i \varepsilon_{str}\, x^t_{(i)}\, \mathrm{F}^r_{(i)} \tag{6.6}$$

is called the **moment of the system of forces**, $\mathrm{F}^r_{(i)}$ or the **torque**. The equation of motion of the system is therefore

$$\frac{d\mathrm{A}_s}{dt} = \mathrm{L}_s . \tag{6.7}$$

Ex. (6.1) (a) Calculate the instantaneous angular momentum of the system in Ex. (5.6). (b) Calculate the instantaneous moment of the forces, given that $\mathrm{F}^i_{(1)} = (4, -5, 2)$, $\mathrm{F}^i_{(2)} = (2, 6, 5)$.

Ans. (a) $$\mathrm{A}_s = \sum_{i=1}^{2} m_{(i)}\, \varepsilon_{str}\, x^t_{(i)}\, \mathrm{v}^r_{(i)}$$
$$= 3(-835, -791, -928) + 5(-99, 167, -192)$$
$$= (-3000, -3208, -3744) .$$

(b)

$$\mathrm{L}_s = \sum_{i=1}^{2} \varepsilon_{str}\, x^t_{(i)}\, \mathrm{F}^r_{(i)} = (46, 66, 79) + (9, 37, -48) = (52, 103, 31) .$$

The equations (6.3) and (6.7) are the equations of motion of the system of particles, whether constituting a rigid body or not. Let us now further specialize the equations to the case of a rigid body. The first equation requires no modification, for it governs the motion of a point, the center of gravity. To particularize equation (6.7), however, we substitute equation (6.1) into equation (6.5), obtaining

$$A^s = \sum_i m_{(i)} \varepsilon^{str} (\xi_t + y_t^{(i)}) (\dot{\xi}_r + \varepsilon_{rpq} \omega^p y_{(i)}^q)$$

$$= \sum_i m_{(i)} \varepsilon^{str} \left[\xi_t \dot{\xi}_r + \xi_t \varepsilon_{rpq} \omega^p y_{(i)}^q + y_t^{(i)} \dot{\xi}_r + y_t^{(i)} \varepsilon_{rpq} \omega^p y_{(i)}^q \right]$$

$$= m \left[\varepsilon^{str} \xi_t \dot{\xi}_r + \varepsilon^{str} \varepsilon_{pqr} \xi_t \omega^p \left(\sum_i m_{(i)} y_{(i)}^q \right) + \varepsilon^{str} \dot{\xi}_r \left(\sum_i m_{(i)} y_t^{(i)} \right) \right.$$

$$\left. + \varepsilon^{str} \varepsilon_{rpq} \omega^p \left(\sum_i m_{(i)} y_t^{(i)} y_{(i)}^q \right) \right].$$

The first term on the right hand side is the angular momentum of the center of gravity with respect to the origin. The second and third terms vanish, by equation (5.2). In considering the final term, we first define the quantity

$$\mathrm{I}_t^q = \sum_i m_{(i)} y_t^{(i)} y_{(i)}^q , \tag{6.8}$$

called the **inertia tensor** of the system about the center of gravity. In terms of this, the last term becomes

$$\varepsilon_{pqr} \varepsilon^{str} \omega^p \mathrm{I}_t^q = \delta_{pqr}^{str} \omega^p \mathrm{I}_t^q = \delta_{pq}^{st} \omega^p \mathrm{I}_t^q$$

$$= \omega^s \mathrm{I}_t^t - \omega^t \mathrm{I}_t^s = \mathrm{I}_p^p \omega^t \delta_t^s - \omega^t \mathrm{I}_t^s = \left(\mathrm{I}\, \delta_t^s - \mathrm{I}_t^s \right) \omega^t ,$$

where

$$\mathrm{I} = \mathrm{I}_p^p \tag{6.9}$$

is called the **inertia invariant**.

We thus have, after lowering indices, that

$$A_s = m \varepsilon_{str} \xi^t \dot{\xi}^r + \left(\mathrm{I}\, \delta_s^t - \mathrm{I}_s^t \right) \omega_t = m \varepsilon_{str} \xi^t \dot{\xi}^r + A_s^{(0)} \tag{6.10}$$

where

$$A_s^{(0)} = \left(\mathrm{I}\, \delta_s^t - \mathrm{I}_s^t \right) \omega_t . \tag{6.11}$$

Equations (6.7), (6.8), (6.9), and (6.10) now describe the motion of a **rigid body.**

As a next step, we may re-fashion the expression (6.6) for the moment of forces. Thus

$$\text{(6.12)}\qquad \begin{aligned} L_s &= \sum_i \varepsilon_{str} x^t_{(i)} F^r_{(i)} = \sum_i \varepsilon_{str}\left(x^t + y^t_{(i)}\right)F^r_{(i)} \\ &= \varepsilon_{str}\xi^t \sum_i F^r_{(i)} + \varepsilon_{str}\sum_i y^t_{(i)} F^r_{(i)} = \varepsilon_{str} x^t F^r + L^{(0)}_s , \end{aligned}$$

where

$$\text{(6.13)}\qquad L^{(0)}_s = \varepsilon_{str}\left(\sum_i y^t_{(i)} F^r_{(i)} \right)$$

is the **moment of forces about the center of gravity**.

The equation of motion (6.7) for a rigid body now becomes

$$\frac{dA_s}{dt} = m\,\varepsilon_{str} x^t \frac{d\dot{\xi}^r}{dt} + \frac{d}{dt}\left[\left(I\,\delta^t_s - I^t_s\right)\omega\right] = \varepsilon_{str}\xi^t F^r + L^{(0)}_s = L_s .$$

Equating the middle terms and using equation (6.2), we see that the equation of motion finally reduces to

$$\text{(6.14)}\qquad \frac{dA^{(0)}_s}{dt} = \frac{d}{dt}\left[\left(I\,\delta^t_s - I^t_s\right)\omega_t\right] = \varepsilon_{str}\sum_i y^t_{(i)} F^r_{(i)} + L^{(0)}_s .$$

We thus see that not only does the center of gravity move as would a free particle of the same total mass, subject to the same set of forces, but at the same time the body moves relative to the center of gravity as though the center of gravity were at rest and the body were subjected to the action of the same forces.

Ex. (6.2) Assume that the two particles of Ex. (5.6) are rigidly attached. (a) Find their inertia tensor. (b) Find their inertia invariant.

Ans.

(a) $$I^q_t = \left\Vert \begin{matrix} 120 & -360 & 240 \\ -360 & 1080 & -720 \\ 240 & -720 & 480 \end{matrix} \right\Vert ;$$

(b) $I = 1680 .$

Ex. (6.3) (a) Calculate the angular momentum of the center of gravity of the particles in Ex. (5.6). (b) Calculate the particles' angular momentum about the center of gravity, using equation (6.11). (c) Determine the total angular momentum and check it against the results of Ex. (6.1a).

Ans. (a) $A^*_s = (-240, 272, 96) .$

(b) $A^{(0)}_s = (-2760, -3480, -3840) .$

7. Rotating Axes

In making use of the equations of motion of a rigid body, it is clear that there will in general occur time derivatives of the inertia tensor and the inertia invariant. Since the inertia tensor characterizes a rigid body, it should be possible to avoid any consideration of the derivatives of the inertia tensor by transforming to moving axes with respect to which the moments of inertia are constant in time.

To this end, consider a rectilinear coordinate system $\bar{y}^r$ which has a fixed orientation in space, and let y^r be the coordinates in a coordinate system rotating about the center of gravity in some way but fixed with respect to the rigid body. Connecting these two systems there will be a transformation such as

$$\bar{y}^r = a_s^r y^s \tag{7.1}$$

where the coefficients a_s^r are functions of time only. Now the distance of any point P from the center of gravity must be the same whether reckoned in the stationary or moving coordinates. Hence

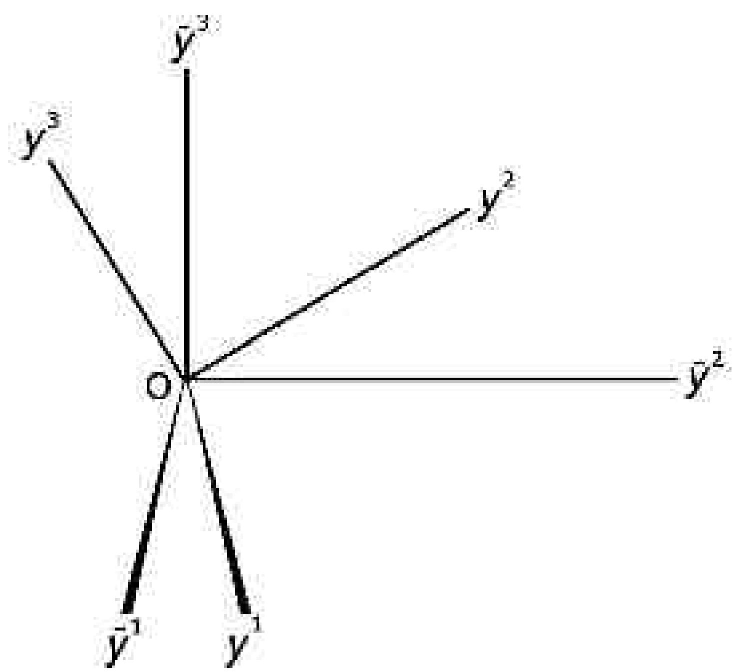
$$g_{mn}\bar{y}^m\bar{y}^n = g_{mn}a_p^m a_q^n y^p y^q = g_{pq}y^p y^q$$

for the coordinates y^p of any point P. Therefore

$$g_{mn}a_p^m a_q^n = g_{pq}.$$

Multiplying both sides by g^{sp} gives

$$g^{sp}\left(g_{mn}a_p^m\right)a_q^n = \left(g^{sp}a_{np}\right)a_q^n = a_n^s a_q^n = g^{sp}g_{pq} = \delta_q^s. \tag{7.2}$$



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From this we see, by taking determinants of both sides, that

$$\left|a_n^s a_q^n\right| = \left|a_n^s\right| \cdot \left|a_q^n\right| = \left|a_n^s\right|^2 = \left|\delta_q^s\right| = 1 .$$

Hence $|a_n^s| = \pm 1$, but since the transformation is a continuous one and we may, without loss of generality, take the initial value of s_n^s to be δ_n^s, we see that the plus sign holds. It is easy to show also that the a_{ns} are, in fact, the cosines of the angles which each of the $\bar{y}^s$ – axes makes at any instant t with the respective y^n – axes.
(See Appendix 3.2).

A further condition upon the a_n^s may be derived by considering a point y^r which is fixed in the rotating system. Then $\dot{y}^r = 0$. Consequently, when we differentiate equation (7.1) we get

$$\frac{d\bar{y}^r}{dt} = \dot{a}_s^r y^s + a_s^r \dot{y}^s = \dot{a}_s^r y^s . \tag{7.3}$$

Since we must also have that

$$\frac{d\bar{y}^r}{dt} = \bar{w}^r = \omega_s^r \bar{y}^s ,$$

equation (7.3) becomes

$$\omega_s^r \bar{y}^s = \dot{a}_s^r y^s .$$

Multiply this by a_r^p; then

$$a_r^p \frac{d\bar{y}^r}{dt} = a_r^p \omega^r \bar{y}^s = a_r^p \dot{a}_s^r y^s .$$

But from equation (7.2) it follows that

$$a_r^p \dot{a}_s^r = - \dot{a}_r^p a_s^r .$$

Therefore, for any y^n or $\bar{y}^s$

$$a_r^p \omega_s^r \bar{y}^s = a_r^p \omega_s^r a_n^s y^n = - \dot{a}_r^p a_s^r y^s = - \dot{a}_s^p a_n^s y^n ,$$

$$\left(a_r^p \omega_s^r + \dot{a}_s^p\right)\bar{y}^s = 0 ,$$

whence

$$- \dot{a}_s^p = a_r^p \omega_s^r . \tag{7.4}$$

We may make use of this result in finding the derivative of the angular momentum. Thus, from equation (7.2) we have that a_s^r is its own inverse, so that the vector transformation from y^r to $\bar{y}^r$ requires that

$$\mathrm{A}_r^{(0)} = a_r^{\ s}\,\bar{\mathrm{A}}_s^{(0)},$$

$$\frac{d\mathrm{A}_r^{(0)}}{dt} = \dot{a}_r^{\ s}\,\bar{\mathrm{A}}_s^{(0)} = a_r^{\ s}\,\frac{d\mathrm{A}_s^{(0)}}{dt} = -\,a_m^{\ s}\,\omega_r^{\ m}\,\bar{\mathrm{A}}_s^{(0)} + a_r^{\ s}\,\frac{d\bar{\mathrm{A}}_s^{(0)}}{dt}$$

$$= -\,\omega_r^{\ m}\,\mathrm{A}_m^{(0)} + \frac{d\bar{\mathrm{A}}_s^{(0)}}{dt},$$

$$\frac{d\mathrm{A}_r^{(0)}}{dt} + \omega_r^{\ m}\,\mathrm{A}_m^{(0)} = a_r^{\ s}\,\frac{d\mathrm{A}_s^{(0)}}{dt}.$$

Now the equation of motion about the center of gravity is

$$\frac{d\bar{\mathrm{A}}_s^{(0)}}{dt} = \bar{\mathrm{L}}_s^{(0)},$$

so that the right hand side of the preceding equation is

$$a_r^{\ s}\,\frac{d\bar{\mathrm{A}}_s^{(0)}}{dt} = a_r^{\ s}\,\bar{\mathrm{L}}_s^{(0)} = \mathrm{L}_r^{(0)}.$$

The final form of the equation in the y^r–coordinates therefore becomes

(7.5) $$\frac{d\mathrm{A}_r^{(0)}}{dt} + \omega_r^{\ m}\,\mathrm{A}_m^{(0)} = \mathrm{L}_r^{(0)}.$$

This is therefore the equation of motion of a rigid body in a system of coordinates with origin at the center of gravity and rotating axes fixed in the body itself.

Let us consider further the inertia tensor I^{rs}. From its definition as

$$\mathrm{I}^{rs} = \sum_i m_{(i)}\,y_{(i)}^r\,y_{(i)}^s$$

it is clear that I^{rs} is symmetric in r and s. Moreover, it is clear that

$$\mathrm{I}_{rs}\,y_{(i)}^r\,y_{(i)}^s = g_{rp}\,g_{sq}\,\mathrm{I}^{pq}\,y_{(i)}^r\,y_{(i)}^s = \sum_i m_{(i)}\left(g_{rp}\,y_{(i)}^r\,y_{(i)}^p\right)\left(g_{sq}\,y_{(i)}^s\,y_{(i)}^q\right)$$

$$= \sum_i m_{(i)}\left(g_{rp}\,y_{(i)}^r\,y_{(i)}^p\right)^2 > 0\,;$$

it is therefore a positive definite form. As we have seen, this implies that the roots of the determinantal equation

$$|\,\mathrm{I}^{rs} - \lambda\,g^{rs}\,| = 0$$

are both real and positive. There is therefore a transformation which will reduce I^{rs} to the diagonal form

(7.6) $$\mathrm{I}^{rs} = \mathrm{I}_{(r)}\,g^{rs}, \quad \mathrm{I}_{(r)} > 0$$

with $\lambda_{(r)} = I_{(r)}$. Moreover, the coordinate system in which I^{rs} has the form of equation (7.6) is an orthogonal system with axes in the invariant directions. This is called a **principal-axis coordinate system**.

If a principal-axis coordinate system is used, we will have for the angular momentum

$$A_s^{(0)} = \left(I\,\delta_s^p - I_{(p)}\,\delta_s^p\right)\omega_p = \left(I - I_{(s)}\right)\omega_s .$$

Equation (7.5) therefore becomes

$$\begin{aligned}&\left(I - I_{(r)}\right)\dot{\omega}_r + \omega_{rm}\left(I - I_{(m)}\right)\omega^m \\ (7.7)\qquad &= \left(I - I_{(r)}\right)\dot{\omega}^r + \varepsilon_{rmn}\,\omega^m\,\omega^n\left(I - I_{(m)}\right) = L_r^{(0)} .\end{aligned}$$

At first glance, the second term on the left hand side of equation (7.7) may appear to be zero because of the term $\varepsilon_{rmn}\,\omega^m\,\omega^n$. However, closer inspection shows that this is in general not the case because of the factor $\left(I - I_{(m)}\right)$. Let us write out the components of this term explicitly; they are

$$\varepsilon_{rmn}\,\omega^m\,\omega^n\left(I - I_{(m)}\right)$$
$$= \sqrt{g}\left[\left(I_{(3)} - I_{(2)}\right)\omega^2\,\omega^3 , \left(I_{(1)} - I_{(3)}\right)\omega^1\,\omega^3 , \left(I_{(2)} - I_{(1)}\right)\omega^1\,\omega^2\right].$$

It is apparent, therefore, that equation (7.7) may be given the final form

$$(7.8)\qquad \left(I - I_{(r)}\right)\dot{\omega}_r - \varepsilon_{rmn}\,I_{(m)}\,\omega^m\,\omega^n = L_r^{(0)} .$$

In practice, this is a somewhat more convenient and useful form than Eq. (7.5).

Ex. (7.1) Show without writing out the terms that $\varepsilon_{rmn}\,I_{(m)}\,\omega^m\,\omega^n$ *does not vanish identically if* $I_{(m)} \neq I_{(n)}$. *(Hint: use the fact that in general* $I_{(m)}\,\omega^m \neq \omega^m$ *.)*

Ex. (7.2) Show that

$$\varepsilon_{rmn}\left(I - I_{(m)}\right)\omega^m\,\omega^n = \varepsilon_{rmn}\,I_{(m)}\,\omega^m\,\omega^n .$$

(Hint: use the result of Ex. (7.1).)

Ex. (7.3) From Appendix (3.1) show that

$$y_i = \bar{y}^1\cos\theta_{i\bar{1}} + \bar{y}^2\cos\theta_{i\bar{2}} + \bar{y}^3\cos\theta_{i\bar{3}} ,$$

hence that $\bar{a}_{i\bar{j}} = \cos\theta_{i\bar{j}}$.

8. Strain

The motion of rigid bodies has been analyzed by considering those motions under which the mutual distances of particles or mass elements remain constant. If this condition be waived, the matter constitutes an elastic medium or fluid. Let us see how such a medium may be described.

Again, let $y^r_{(i)}$ be the position vector of a mass element. The distance between any two elements i and j is then

$$\Delta_{(ij)} = \left[g_{rs} \left(y^r_{(i)} - y^r_{(j)} \right) \left(y^s_{(i)} - y^s_{(j)} \right) \right]^{1/2}.$$

Suppose each element to be continuously displaced during a small time interval dt. Then the separation of elemenmattts i and j changes at a rate

(8.1) $$\frac{d\Delta_{(ij)}}{dt} = \frac{g_{rs}}{\Delta_{(ij)}} \left(y^r_{(i)} - y^r_{(j)} \right) \left(\frac{dy^s_{(i)}}{dt} - \frac{dy^s_{(j)}}{dt} \right).$$

Now if both mass elements are near the origin and if dt is small, then to a first approximation (by a Maclaurin series, for example)

(8.2) $$\frac{dy^r_{(i)}}{dt} = a^r + b^{rs} y^{(i)}_s,$$

terms of higher order being neglected. Here b^{rs} is taken to be independent of $y^k_{(i)}$; so long as $y^s_{(i)}$ is small, there is no loss of generality in this assumption.

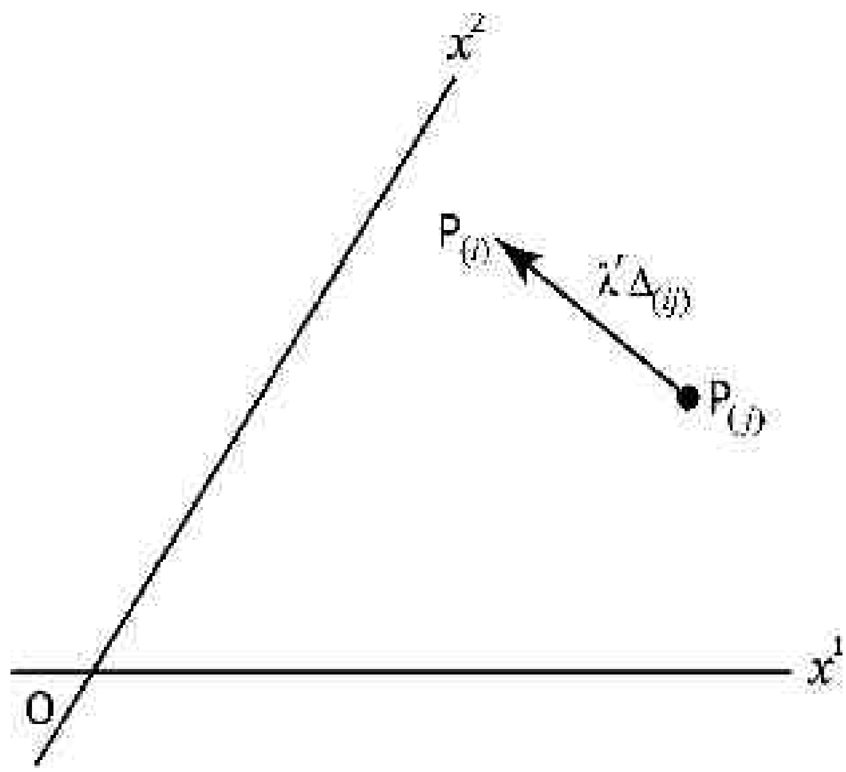

Figure 69

The vector a^r clearly contributes nothing to the right hand side of equation (8.1); it corresponds to a pure translation. To assess the contribution of b^{rs}, we resolve it into symmetric and antisymmetric parts; thus

$$b^{rs} = \frac{1}{2}\left(b^{rs} + b^{sr}\right) + \frac{1}{2}\left(b^{rs} - b^{sr}\right) = e^{rs} + \omega^{rs}. \tag{8.3}$$

Then

$$\frac{d\Delta_{(ij)}}{dt} = \frac{g_{rs}}{\Delta_{(ij)}}\left(y^r_{(i)} - y^r_{(j)}\right) b^{sp}\left(y_p^{(i)} - y_p^{(j)}\right)$$

$$= \frac{b_{sp}}{\Delta_{(ij)}}\left(y^s_{(i)} - y^s_{(j)}\right)\left(y^p_{(i)} - y^p_{(j)}\right)$$

$$= \frac{e_{sp}}{\Delta_{(ij)}}\left(y^s_{(i)} - y^p_{(j)}\right) + \frac{\omega_{sp}}{\Delta_{(ij)}}\left(y^s_{(i)} - y^s_{(j)}\right)\left(y^p_{(i)} - y^p_{(j)}\right).$$

The last term vanishes inasmuch as ω_{sp} is antisymmetric in s and p; it corresponds to a pure rigid rotation.

Both translation and rotation are rigid body displacements. The sole remaining term, the only one making a non-zero contribution to the variation of the separation of the elements i and j, is

$$\frac{d\Delta_{(ij)}}{dt} = \frac{e_{rs}}{\Delta_{(ij)}}\left(y^r_{(i)} - y^r_{(j)}\right)\left(y^s_{(i)} - y^s_{(j)}\right). \tag{8.4}$$

The symmetric tensor e_{rs} is called the **rate of strain tensor**. It fully characterizes the manner in which the medium has been distorted at each point.

To interpret the strain tensor, consider two points $P_{(i)}$ and $P_{(j)}$ (Fig. 69) between which the vector is initially

$$y^r_{(i)} - y^r_{(j)} = \lambda^r \Delta_{(ij)},$$

where λ^r is clearly a unit vector in the direction from $P_{(j)}$ to $P_{(i)}$. Then the strain e_{rs} will be such that the points $P_{(i)}$ and $P_{(j)}$ separate at a rate

$$\frac{d\Delta_{(ij)}}{dt} = \Delta_{(ij)}\, e_{rs}\, \lambda^r \lambda^s.$$

In other words, the distance between elements i and j increases in proportion to the initial distance $\Delta_{(ij)}$ between them and to $e_{rs}\lambda^r\lambda^s$. Since the unit vector λ^r may be in any direction whatever, depending on the identity of the elements, it is clear that e_{rs} completely characterizes the distortion of the medium in the neighborhood of every point. Thus, to specify a distortion is to determine a strain tensor and vice versa.

Appendix 3.1 : The Transformation to New Axes and New Units of Length

Consider three planes which mutually intersect by pairs and have the single point O in common. We take the point O as the origin (see Fig. 70). Let the intersections of the planes be the lines OX^1, OX^2, OX^3. They will serve as coordinate axes. Through any point P draw a line PB parallel to OX^3. Point B is the

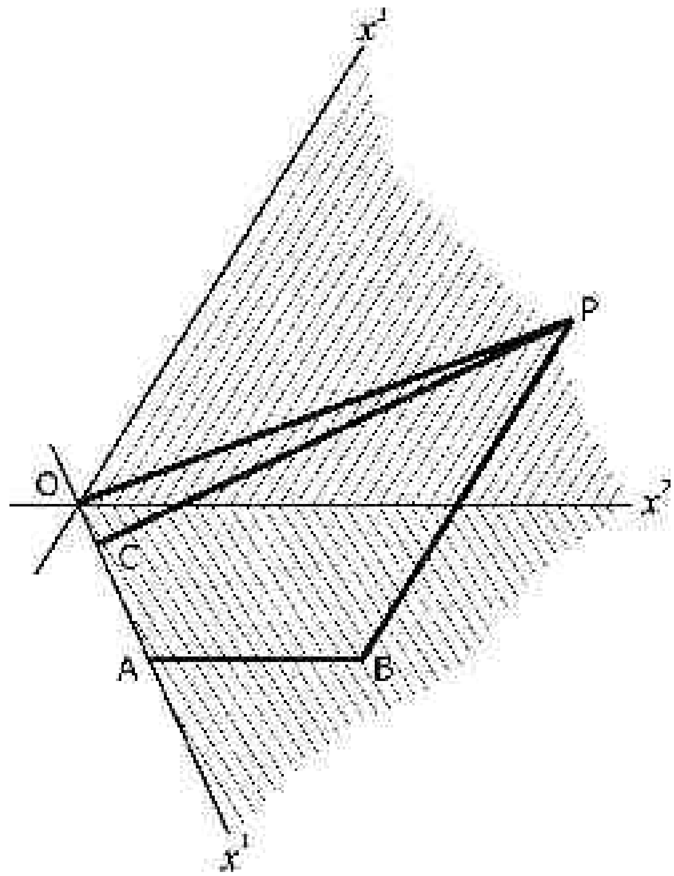

Figure 70

intersection of this line with the plane of OX^1 and OX^2. Through B draw a line parallel to OX^2. This line intersects the axis OX^1 at A. Then by definition, the rectilinear contravariant coordinates of the point P are

$$x^1 = \overline{OA}, \; x^2 = \overline{AB}, \; x^3 = \overline{BP}.$$

At the same time, we project OP onto $x^1 = \overline{OA}$, $x^2 = \overline{AB}$, $x^3 = \overline{BP}$. If the perpendicular onto OX^1 intercepts the length $\overline{OC}$, then the covariant rectilinear coordinate x_1 is $x_1 = \overline{OC}$ in this system of axes. Projections onto the other axes give the other covariant rectilinear coordinates.

Now the projection of any continuous set of line segments beginning at O and ending at P is the same as the projection of $\overline{OP}$ itself. Thus the projection of the jointed curve $\overline{OABP}$ is the same as the projection of $\overline{OP}$ upon the same axes; it is the sum of the projections of $\overline{OA}$, $\overline{AB}$, and $\overline{BP}$.

Hence

$$\overline{\mathrm{OP}}\cos(\mathrm{AOP}) \equiv x_1 = \overline{\mathrm{OA}} + \overline{\mathrm{AB}}\cos(\mathrm{X}^1\mathrm{X}^2) + \overline{\mathrm{BP}}\cos(\mathrm{X}^1\mathrm{X}^3)$$

or

$$x_1 = x^1 + x^2\cos\theta_{12} + x^3\cos\theta_{13},$$

where we have set $\theta_{12} = \mathrm{X}^1\mathrm{O}\mathrm{X}^2$, $\theta_{13} = \mathrm{X}^1\mathrm{O}\mathrm{X}^3$, etc., and used the fact that $\theta_{11} = 0$. A similar result holds for any choice of axes. Therefore in any coordinate system,

(A3.1.1)
$$x_j = \sum_k x^k\cos\theta_{jk}$$

relating covariant and contravariant coordinates of a given point in the same rectilinear system.

The inverse transformation of the first set may easily be found to be

(A3.1.2)
$$\begin{cases} x^1 = \dfrac{\sin\theta_{23}}{g}\Big[\big(\sin\theta_{23}\big)x_1 - \big(\sin\theta_{13}\cos\alpha_{132}\big)x_2 - \big(\sin\theta_{12}\cos\alpha_{123}\big)x_3\Big] \\[2ex] x^2 = \dfrac{\sin\theta_{13}}{g}\Big[-\big(\sin\theta_{23}\cos\alpha_{132}\big)x_1 + \big(\sin\theta_{13}\big)x_2 - \big(\sin\theta_{12}\cos\alpha_{213}\big)x_3\Big], \\[2ex] x^3 = \dfrac{\sin\theta_{12}}{g}\Big[-\big(\sin\theta_{23}\cos\alpha_{123}\big)x_1 - \big(\sin\theta_{13}\cos\alpha_{213}\big)x_2 + \big(\sin\theta_{12}\big)x_3\Big], \end{cases}$$

where g is the common value of

(A3.1.3)
$$\begin{aligned} g &= 1 + 2\cos\theta_{12}\cos\theta_{23}\cos\theta_{13} - \cos^2\theta_{12} - \cos^2\theta_{13} - \cos^2\theta_{23} \\ &= \big[\sin\theta_{12}\sin\theta_{13}\sin\alpha_{213}\big]^2 = \big[\sin\theta_{23}\sin\theta_{12}\sin\alpha_{123}\big]^2 \\ &= \big[\sin\theta_{13}\sin\theta_{23}\sin\alpha_{132}\big]^2 \end{aligned}$$

and α_{ijk} is the dihedral angle along the line OX^j between the (x^i, x^j)-plane and the (x^j, x^k)-plane.

In equation (A3.1.1) we recognize the three-dimensional form of $x_i = g_{ij}x^i$, whence we identify g_{ij} as

(A3.1.4)
$$g_{ij} = \begin{Vmatrix} 1 & \cos\theta_{12} & \cos\theta_{13} \\ \cos\theta_{12} & 1 & \cos\theta_{23} \\ \cos\theta_{13} & \cos\theta_{23} & 1 \end{Vmatrix}$$

when the scale factors are $s_{(1)} = s_{(2)} = s_{(3)} = 1$. At the same time, we see in equation (A3.1.2) the three-dimensional form of $x^i = g^{ij} x_j$, whence we identify g^{ij} as

(A3.1.5)
$$\begin{cases} g^{11} = \dfrac{1}{\sin^2\theta_{12}\sin^2\alpha_{123}} = \dfrac{1}{\sin^2\theta_{13}\sin^2\alpha_{132}}, \\ g^{12} = -\dfrac{\cot\alpha_{132}}{\sin\theta_{13}\sin\theta_{23}\sin\alpha_{132}} = g^{21}, \\ g^{13} = -\dfrac{\cot\alpha_{123}}{\sin\theta_{12}\sin\theta_{23}\sin\alpha_{132}} = g^{31}, \\ g^{22} = \dfrac{1}{\sin^2\theta_{12}\sin^2\alpha_{213}} = \dfrac{1}{\sin^2\theta_{23}\sin^2\alpha_{132}}, \\ g^{23} = -\dfrac{\cot\alpha_{123}}{\sin\theta_{12}\sin\theta_{23}\sin\alpha_{123}} = g^{32}, \\ g^{33} = \dfrac{1}{\sin^2\theta_{13}\sin^2\alpha_{213}} = \dfrac{1}{\sin^2\theta_{23}\sin^2\alpha_{123}}. \end{cases}$$

Let us now seek to determine the effects of a transformation from an x^i-coordinate system to an $\bar{x}^i$-coordinate system where

$$\bar{x}^i = \bar{a}_j^{\,i} x^j, \; x^j = a_k^j \bar{x}^k, \; \bar{a}_j^{\,i} a_k^j = \delta_k^i .$$

Let us begin with the fact that

$$\bar{g}_{ij} = \bar{s}_{(i)}\bar{s}_{(j)}\cos\theta_{\overline{ij}} = \frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j} g_{mn} = \sum_{m,n} a_i^{\,m} a_j^{\,n} s_{(m)} s_{(n)} \cos\theta_{mn}$$

When $i = j$, we have

$$\bar{g}_{ii} = \bar{s}_{(i)}^{\;2} .$$

From this we get

$$\cos\theta_{\overline{ij}} = \frac{\bar{g}_{ij}}{\sqrt{\bar{g}_{ii}\bar{g}_{jj}}} = \frac{\sum_{m,n} a_k^{\,m} a_l^{\,n} s_{(m)} s_{(n)} \cos\theta_{mn}}{\sqrt{\bar{g}_{ii}\bar{g}_{jj}}},$$

wherewith the angles $\theta_{\bar{i}\bar{j}}$ between the new axes are determined from the coefficients of the transformation and the known scale factors and angles $s_{(i)}$ and θ_{ij}, respectively.

As an example, consider a transformation from a Cartesian system of coordinates x^i to a new system where

$$x^1 = a_j^1 \bar{x}^j = 3\bar{x}^1 - \bar{x}^2 + 5\bar{x}^3 ,$$
$$x^2 = a_j^2 \bar{x}^j = 2\bar{x}^1 + 4\bar{x}^2 - \bar{x}^3 ,$$
$$x^3 = a_j^3 \bar{x}^j = 4\bar{x}^1 - 2\bar{x}^2 + 3\bar{x}^3 .$$

Then

$$\bar{g}_{11} = 29 ,\ \bar{g}_{12} = -3 ,\ \bar{g}_{13} = 25 ,$$
$$\bar{g}_{22} = 21 ,\ \bar{g}_{23} = -15 ,\ \bar{g}_{33} = 35 .$$

Therefore

$$\bar{s}_{(1)} = \sqrt{\bar{g}_{11}} = \sqrt{29} ,\ \bar{s}_{(2)} = \sqrt{21} ,\ \bar{s}_{(3)} = \sqrt{\bar{g}_{33}} = \sqrt{35} ,$$

$$\cos\theta_{\bar{1}\bar{2}} = \frac{g_{\bar{1}\bar{2}}}{\sqrt{\bar{g}_{\bar{1}\bar{1}}\,\bar{g}_{\bar{2}\bar{2}}}} = -\frac{3}{\sqrt{29\times 21}} = -0.12157 ,\ \theta_{\bar{1}\bar{2}} = 96^\circ\, 58'\, 57'' ,$$

$$\cos\theta_{\bar{1}\bar{3}} = \frac{\bar{g}_{\bar{1}\bar{3}}}{\sqrt{\bar{g}_{\bar{1}\bar{1}}\times\bar{g}_{\bar{3}\bar{3}}}} = \frac{25}{\sqrt{29\times 35}} = 0.78471 ,\ \theta_{\bar{1}\bar{3}} = 38^\circ\, 18'\, 23'' ,$$

$$\cos\theta_{\bar{2}\bar{3}} = \frac{\bar{g}_{\bar{2}\bar{3}}}{\sqrt{\bar{g}_{\bar{2}\bar{2}}\times\bar{g}_{\bar{3}\bar{3}}}} = -\frac{15}{\sqrt{21\times 35}} = -0.55328 ,\ \theta_{\bar{2}\bar{3}} = 123^\circ\, 35'\, 33'' .$$

To find the angles which the new axes make with the old, we first define unit vectors along the respective axes. Thus

$$\lambda^i_{(1)} = (1, 0, 0) ,\ \lambda^i_{(2)} = (0, 1, 0) ,\ \text{and}\ \lambda^i_{(3)} = (0, 0, 1) ;$$

$$\bar{\lambda}_{(\bar{1})} = \left(\frac{1}{\sqrt{\bar{g}_{\bar{1}\bar{1}}}}, 0, 0\right) ,\ \bar{\lambda}_{(\bar{2})} = \left(0, \frac{1}{\sqrt{\bar{g}_{\bar{2}\bar{2}}}}, 0\right) ,\ \bar{\lambda}_{(\bar{3})} = \left(0, 0, \frac{1}{\sqrt{\bar{g}_{\bar{3}\bar{3}}}}\right) .$$

We now transform the $\bar{\lambda}^i_{(\bar{j})}$ to the Cartesian system by the equation of vector transformation, giving

$$\lambda^i_{(\bar{j})} = \frac{\partial x^i}{\partial \bar{x}^k}\bar{\lambda}^k_{(\bar{j})} = a_k^i \bar{\lambda}^k_{(\bar{j})} ,$$

$$\lambda^i_{(\bar{1})} = \frac{1}{\sqrt{29}}(3, 2, 4) ,$$

$$\lambda^i_{(\bar{2})} = \frac{1}{\sqrt{21}}(-1, 4, -2) ,$$

$$\lambda^i_{(\bar{3})} = \frac{1}{\sqrt{35}}(5, -1, 3) .$$

The angle between these unit vectors and those along the Cartesian axes is

$$\theta_{i\bar{j}} = \cos^{-1} g_{mn} \lambda_i^m \lambda_{\bar{j}}^j$$

Hence

$$\theta_{1\bar{1}} = \cos^{-1} \frac{3}{\sqrt{29}} = 56° \, 8' \, 44''$$

$$\theta_{1\bar{2}} = \cos^{-1}\left(-\frac{1}{\sqrt{21}} \right) = 102° \, 36' \, 16'',$$

$$\theta_{1\bar{3}} = \cos^{-1} \frac{5}{\sqrt{35}} = 32° \, 18' \, 42''.$$

The angles between the other axes could be determined in a similar way, but the results would be redundant inasmuch as the three scale factors and six angles completely determine the relation of the two cordinate systems.

Appendix 3.2: Interpretation of the Transformation to Rotating Axes

We may readily interpret the coefficients of $a_s^{\ r}$ by considering what operation the tensor $a_s^{\ r}$ performs upon a unit vector along an axis. Thus, let

$$\lambda_{(j)}^i = e_{(j)}^i, \; \lambda_i^{(j)} = g_{ik} \lambda_{(j)}^k = g_{ij}$$

be such a unit vector along the $j-$axis. Then the vector

$$\lambda_{(j)}^r = a_s^{\ r} \lambda_{(j)}^2$$

is one whose components are

$$\bar{\lambda}_{(j)}^r = a_s^{\ r} \lambda_{(j)}^2 = (a_j^{\ 1}, a_j^{\ 2}, a_j^{\ 3}).$$

Let us form the inner product of $\bar{\lambda}_{(j)}^r$ with $\lambda_r^{(k)}$.

Then

$$\bar{\lambda}_{(j)}^r \lambda_r^{(k)} = | \bar{\lambda}_{(j)}^r | \cdot | \lambda_r^{(k)} | \cdot \cos \theta_{\bar{j}k} = | \lambda_{(j)}^r | \cdot 1 \cdot \cos \theta_{\bar{j}k} = a_j^{\ r} g_{rk} = a_{jk},$$

where $\theta_{\bar{j}k}$ is the angle between $\bar{\lambda}_{(j)}^r$ and $\lambda_{(k)}^r$. If we now require that $\bar{\lambda}_{(j)}^r$ also be a unit vector, we see that

$$\cos \theta_{\bar{j}k} = a_{jk}.$$

We have only to adopt the $\bar{\lambda}_{(j)}^r$ as a unit vector along the $\bar{y}^j-$axes to see that the transformation (7.1) converts the $\bar{y}^j-$axes into the y^j-axes. Therefore the a_{ij} are the cosines of the angles between the two sets of axes, hence symmetric.

Notes — Chapter 3

§3.3. For a more extensive treatment of points, lines and planes, see McConnell (9), Chs. IV and V.

§3.4. For a more extensive treatment of cones and quadrics, see McConnell (9), Chs. VI — IX.

§§3.5 — 3.7. Many common problems concerning systems of particles and the mechanics of rigid bodies are considered in detail in Spiegel (16), Chs. 7 to 10.

§3.5. In general, equation (5.11) is implied by equation (5.10). This may be seen by noting that since i and j are arbitrary, and because in general we can elect to have the y^r-axis pass through particle j and the y^s-axis pass through particle i, the coordinates may be chosen so that $y^r_{(j)} = a \neq 0$, $y^s_{(j)} = 0$, $y^s_{(i)} = b \neq 0$, $y^r_{(i)} = 0$, where r and s have fixed (rather than general) values. Then equation (5.10) states that

$$(\omega_{rs} + \omega_{sr})\, a b = 0, \; (a b \neq 0),$$

whence equation (5.11) follows. The same argument may be applied for every r and s. It clearly fails or is inconclusive when $y^r_{(i)} = 0 = y^r_{(j)}$, i.e., when all particles are in a common coordinate plane or along a common axis.

§3.7. Note that the coefficients a^r_s in equation (7.1) are functions of time but that the coordinates y^s are not. Since the principal–axis system is related to the coordinates y^s by a linear transformation with constant coefficients, the coordinates of the principal–axis system are also time–free.

§3.8. The fact that a^r and b^{rs} are to be independent of $y^r_{(i)}$ corresponds to the assumption that the distortion of the medium is continuous — i.e., without fractures or interfaces.